Real Analysis 1, MATH 5210, Fall 2022

Homework 5, Section 2.3 The σ-Algebra of Lebesgue Measurable Sets

Due Saturday, September 24, at 11:59 p.m.

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

2.13. (Modified) Prove that:

- **(i′)** the translate of an open set is open,
- **(ii′)** the translate of a closed set is closed,
- **(i)** the translate of an F_σ set is also F_σ,
- **(ii)** the translate of a G_δ set is also G_δ, and
- **(iii)** the translate of a set of measure zero also has measure zero.

2.3.A. The symmetric difference of sets A and B is $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Prove that if A is measurable and $m^*(A \Delta B) = 0$ then B is measurable and $m^*(A) = m^*(B)$. HINT: Use the fact that the measurable sets form a σ-algebra (by Theorem 2.9) to write B in terms of measurable sets using unions, intersections, and complements. Use the countable additivity of m^* on measurable sets (Proposition 2.13).

2.3.B. **(a)** Prove that if G is a bounded open set, then $m^*(G)$ equals the sum of the lengths of its "constituent" countable disjoint open intervals (these intervals are the *connected components* of G). HINT: Use Theorem 0.7 to write $G = \cup_{n=1}^\infty I_n$ where each I_n is a bounded open interval and prove $m^*(G) = \sum_{n=1}^\infty \ell(I_n)$.

(b) Prove that if G is an open set, then $m^*(G)$ equals the sum of the lengths of its constituent countable disjoint open intervals. HINT: Consider two cases. First consider the case where each constituent open interval of G is bounded and second consider the case where G has an unbounded constituent open interval.