Real Analysis 1, MATH 5210, Spring 2017

Homework 10, Projections and Hilbert Space Isomorphisms (HWG5.4)

Due Friday, April 7, at 1:40

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

5.4.6. Prove that $R = \{(1,0,0,\ldots),(0,1,0,\ldots),\ldots\} \subset \ell^2$ is a (topologically) closed set and a bounded set, but not a compact set. NOTE: Recall that the Heine-Borel Theorem (Theorem 1.4.11) states that a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. The example given here shows that the familiar Heine-Borel Theorem does not hold in all metric spaces. Also notice that R is an infinite bounded set with no limit points, indicating that the Bolzano-Weierstrass Theorem (see Exercise 3 of Section 1.4) does not hold in ℓ^2.

5.4.8 Prove that if $\{r_1, r_2, \ldots\}$ is an orthonormal basis for Hilbert space H, and for $h \in H$ we have $\langle h, r_i\rangle = 0$ for all $i \in \mathbb{N}$, then $h = 0$.

5.4.10. Apply the Gram-Schmidt process to the set $\{1, x, x^2, x^3\}$ in the space $L^2([-1,1])$ to produce an orthonormal set. Verify that this yields the first four normalized Legendre polynomials (see Exercise 3 of Section 5.3). HINT: The nth normalized Legendre polynomial is give by $\sqrt{n+\frac{1}{2}}\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$ for $n = 0, 1, 2, \ldots$.