Real Analysis 1, MATH 5210, Spring 2017

Homework 6, L^p is Complete:

The Riesz-Fischer Theorem (7.3)

Due Friday, February 24, at 1:30

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

7.25. Let E have finite measure and $1 \leq p_1 < p_2 \leq \infty$. If $\{f_n\} \to f$ in $L^{p_2}(E)$ (i.e., with respect to the L^{p_2} norm) then $\{f_n\} \to f$ in $L^{p_1}(E)$.

7.26. The L^p Lebesgue Dominated Convergence Theorem.
Let $\{f_n\}$ be a sequence of measurable functions that converge pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose there is a function $g \in L^p(E)$ such that for all $n \in \mathbb{N}$, $|f_n| \leq g$ a.e. on E. Then $\{f_n\} \to f$ in $L^p(E)$.

7.34. (a) Let $1 \leq p \leq \infty$. Prove that every rapidly Cauchy sequence in ℓ^p converges with respect to the ℓ^p norm to an element of ℓ^p. HINT: For rapidly Cauchy sequence $\{a_k\}_{k=1}^{\infty}$ of elements of ℓ^p, define sequence T_{a_k} as in Exercise 7.12. Prove that $\{T_{a_k}\}_{k=1}^{\infty}$ is rapidly Cauchy in $L^p([1,\infty))$. Prove that the limit of $\{T_{a_k}\}$ is of the form T_a for some $a \in \ell^p$ (you will need to show that T_a is constant on each $[n, n+1)$ for $n \in \mathbb{N}$). Confirm that $\{a_k\}$ converges to a with respect to the ℓ^p norm.

(b) Prove that the ℓ^p spaces are Banach spaces for $1 \leq p \leq \infty$. This is the Riesz-Fischer Theorem for ℓ^p.