Real Analysis 1, MATH 5210, Spring 2017

Homework 7, Approximation and Separability (7.4); Solutions

Due Friday, March 3, at 1:30

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

7.36. Let $\mathcal{S}$ be a subset of a normed linear space X. Prove that $\mathcal{S}$ is dense in X if and only if each $g \in X$ is the limit of a sequence in $\mathcal{S}$.

7.39. Let E be a measurable set, $1 \leq p < \infty$, q the conjugate of p, and $\mathcal{S}$ a dense subset of $L^q(E)$. Prove that if $g \in L^p(E)$ and $\int_E gf = 0$ for all $f \in \mathcal{S}$, then $g = 0$. HINT: ASSUME $g \neq 0$ (that is, $\|g\|_p \neq 0$). Define g^* as in Hölder's Inequality. Let $\{f_n\} \subset \mathcal{S}$ be a sequence in $\mathcal{S}$ with limit g^* (why does such a sequence exist?). Use Reisz-Fischer to find a subsequence $\{f_{n_k}\}$ which converges to g^* pointwise. Show that $\{gf_{n_k}\} \to gg^*$ in $L^1(E)$. Use Exercise 7.27 with $p = 1$ to find $f \in L^1(E)$ as described and then use the Lebesgue Dominated Convergence Theorem to get a contradiction.

7.44. For $1 \leq p < \infty$, show that ℓ^p is separable. Show that ℓ^∞ is not separable.