Real Analysis 1, MATH 5210, Spring 2019

Homework 10, Measures and Measurable Sets (17.1)

Due Tuesday, April 16, at 2:15

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

17.4. Let $\{(X_\lambda, \mathcal{M}_\lambda, \mu_\lambda)\}_{\lambda \in \Lambda}$ be a collection of measure spaces parametrized by the set Λ. Assume the collection of sets $\{X_\lambda\}_{\lambda \in \Lambda}$ is disjoint. Then we can form a new measure space (called their *union*) $(X, \mathcal{B}, \mu)$ by letting$X = \cup_{\lambda \in \Lambda} X_\lambda$, $\mathcal{B}$ be the collection of subsets B of X such that $B \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \lambda$ and defining $\mu(B) = \sum_{\lambda \in \Lambda} \mu_\lambda(B \cap X_\lambda)$ for $B \in \mathcal{B}$. (If only a countable number of summands is nonzero then we take the usual definition of this sum as a series; if an uncountable number of terms are nonzero then we assign the value ∞.)

(i) Prove that $\mathcal{B}$ is a σ-algebra.

(ii) (Bonus) Prove that μ is a measure.

17.6. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $X_0 \in \mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Prove that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

17.7. Let $(X, \mathcal{M})$ be a measurable space.

(i) If μ and ν are measures defined on $\mathcal{M}$, then the set function λ defined on $\mathcal{M}$ by $\lambda(E) = \mu(E) + \nu(E)$ also is a measure, denoted $\lambda = \mu + \nu$.

17.9. (Bonus) Prove Proposition 17.3: Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets of E of X of the form $E = A \cup B$ where $B \in \mathcal{M}$ and $A \subset C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$. For such a set E define $\mu_0(E) = \mu(B)$. Then $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ, and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space.