Real Analysis 2, MATH 5220, Spring 2023

Homework 7, 8.1. The Riesz Representation for the Dual of L^p, $1 \leq p < \infty$, Solutions

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

8.7(a) State and prove a version of Proposition 8.2 for ℓ^p where $1 \leq p < \infty$.

HINT 1: The statement of the proposition is: "Let $1 \leq p < \infty$, let q be the conjugate of p, and let $b = \{b_k\}_{k=1}^\infty \in \ell^q$. Define the functional T on ℓ^p by $T(a) = \sum_{k=1}^\infty a_k b_k$ for all $a = \{a_k\}_{k=1}^\infty \in \ell^p$. Then T is a bounded linear functional on ℓ^p and $\|T\|_* = \|b\|_q$."

HINT 2: You need Hölder's Inequality for ℓ^p (to be given in Exercise 7.12(b)): "Let $1 \leq p < \infty$, and q the conjugate of p. The for $a = \{a_k\}_{k=1}^\infty \in \ell^p$ and $b = \{b_k\}_{k=1}^\infty \in \ell^q$ we have $(a_k b_k) \in \ell^1$ and $\sum_{k=1}^\infty |a_k b_k| \leq \|a\|_p \|b\|_q$. Moreover, if $b \neq 0$, then $b^* = \{\|b\|_p^{1-p} \mathrm{sgn}(b_k)|b_k|^{p-1}\}_{k=1}^\infty$ if $p > 1$, and $b^* = \{\mathrm{sgn}(b_k)\}_{k=1}^\infty$ if $p = 1$, is an element of ℓ^q, $\sum_{k=1}^\infty b_k^* b_k = \|b\|_p$, and $\|b^*\|_q = 1$."

8.7(c) State and prove a Riesz Representation Theorem for the bounded linear functionals on ℓ^p for $1 \leq p < \infty$.

HINT 1: The statement of the theorem is: "Let $1 \leq p < \infty$ and let q be the conjugate of p. For each $b \in \ell^q$, define the bounded linear functional $\mathcal{R}_b$ on ℓ^p by

$$\mathcal{R}_b(a) = \sum_{k=1}^\infty a_k b_k \text{ for all } a = \{a_k\}_{k=1}^\infty \in \ell^p.$$

The for each bounded linear functional T on ℓ^p, there is a unique $b \in \ell^q$ for which $\mathcal{R}_b = T$ and $\|T\|_* = \|b\|_q$."

HINT 2: You need to assume Exercise 8.7(b) which involves giving a version of Theorem 8.5 for ℓ^p where $1 \leq p < \infty$. The result would state: "Suppose T is a bounded linear functional on ℓ^p. Then there is $b = \{b_k\}_{k=1}^\infty \in \ell^q$ where q is the conjugate of p for which $T(a) = \sum_{k=1}^\infty a_k b_k$ for all $a = \{a_k\}_{k=1}^\infty \in \ell^p$."