Chapter 10. Metric Spaces: Three Fundamental Theorems

Note. In this chapter we consider the Arzelà-Ascoli Theorem (which concerns convergence of a sequence of functions on a compact metric space), the Baire Category Theorem (which concerns countable unions and intersections of open dense sets and closed hollow sets, respectively, in a complete metric space), and the Banach Contraction Principle (sometimes called the "Contraction Mapping Theorem"; it concerns fixed points of contractions on a complete metric space).

Section 10.1. The Arzelà-Ascoli Theorem

Note. We state and prove the Arzelà-Ascoli Theorem, which gives a condition a uniformly bounded sequence of continuous real-valued functions on a compact metric space X to have a uniformly convergent subsequence. We apply this to finding conditions for a subset of a metric space to be compact.

Note. We know that for compact X and continuous real-valued function f on X (i.e., $f \in C(X)$) that, by the Extreme Value Theorem (Theorem 9.22), f attains a maximum value. We define the norm $\|f\|_{\max} = \max_{x\in X} |f(x)|$; this in fact is a norm, as can be shown similar to the special case that it is a norm on $C[a,b]$, as is shown in Problem 7.1). The maximum norm induces a metric on $C(X)$ given by:

$$\rho_{\max}(g,h) = \|g-h\|_{\max} \text{ for all } g,h \in C(X).$$

Definition. The metric on $C(X)$, where X is a compact metric space, is the *uniform metric* on $C(X)$. A sequence of $C(X)$ that is Cauchy with respect to the uniform metric is a *uniformly Cauchy sequence.*

Note. The name "uniform metric" is used for $\rho_{\max}$ since a sequence in $C(X)$ converges with respect to $\rho_{\max}$ if and only if the sequence converges uniformly on X. The proof that $C(X)$ is complete is the same as the proof that $C[a,b]$ is complete (which was given in Section 9.4. Complete Metric Spaces ; see Proposition 9.10). So we state the following without proof.

Proposition 10.1. If X is a compact metric space, then $C(X)$ is complete.

Note. The next definition plays a large role in the Arzelà-Ascoli Theorem.

Definition. A collection $\mathcal{F}$ of real-valued functions on a metric space X is *equicontinuous* at the point $x \in X$ provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every $f \in \mathcal{F}$ and $x' \in X$ we have

$$\text{if } \rho(x', x) < \delta, \text{ then } |f(x') - f(x)| < \varepsilon.$$

The collection $\mathcal{F}$ is *equicontinuous on X* provided it is equicontinuous at every point in X.

Note. In a collection of functions equicontinuous on X, each function is continuous on X. However, a collection of functions can be each continuous on X, while the collection is not equicontinuous on X. For example, consider the continuous functions $f_n(x) = x^n$ on $X = [0,1]$, for each $n \in \mathbb{N}$. Then the collection $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$ is not equicontinuous at $x = 1$ (though it is at each other element of $[0,1]$). So equicontinuity and continuity are different concepts (for a collection of functions).

Example. Let $\mathcal{F}$ be a collection of continuous real-valued functions on $[a,b]$ that are differentiable on (a,b) and there is $M \geq 0$ for which all $f \in \mathcal{F}$ satisfy $|f'(x)| \leq M$ for $x \in (a,b)$. By the Mean Value Theorem, $|f(u) - f(v)| \leq M|u - v|$ for all $u, v \in [a,b]$. Then $\mathcal{F}$ is equicontinuous on X since we can simply take $\delta = \varepsilon/M$ (when $M > 0$) in the ε/δ definition of continuity. This example motivates the following definition.

Definition. A sequence $\{f_n\}$ of real-valued functions on a set X is *pointwise bounded* provided for each $x \in X$, the sequence $\{f_n(x)\}$ is bounded and is *uniformly bounded on* X provided there is some $M \geq 0$ for which $|f_n(x)| \leq M$ on X for all n.

Note. The Arzelà-Ascoli Theorem deals with uniformly convergent subsequence of a uniformly bounded, equicontinuous sequence of real-valued functions on a compact metric space. First we consider a pointwise bounded equicontinuous sequence in $C(X)$ in the following.

Lemma 9.2. The Arzelà-Ascoli Lemma.

Let X be a separable metric space and $\{f_n\}$ an equicontinuous sequence in $C(X)$ that is pointwise bounded. Then a subsequence of $\{f_n\}$ converges pointwise on all of X to a real-valued function f on X.

Definition. A collection $\mathcal{F}$ of real-valued functions on a metric space X is *uniformly equicontinuous* if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for $u, v \in X$ and for an $f \in \mathcal{F}$ we have

$$\text{if } \rho(u, v) < \delta \text{ then } |f(u) - f(v)| < \varepsilon.$$

Note. In Proposition 9.23 of Section 9.5. Compact Metric Spaces we proved that a continuous mapping of a compact metric space to another metric space is uniformly continuous. That proof can be modified to show an equicontinuous family of continuous functions on a compact metric space is uniformly equicontinuous. This is to be done in Problem 10.3. We now state and prove the main result of this section, the Arzelà-Ascoli Theorem. Notice that it concerns the existence of a subsequence of a uniformly bounded equicontinuous sequence of functions on a compact metric space and that the subsequence has not only a uniform limit but that the limit function is continuous.

Theorem 10.1.A. The Arzelà-Ascoli Theorem.

Let X be a compact metric space and $\{f_n\}$ is a uniformly bounded, equicontinuous sequence of real-valued functions on X. Then $\{f_n\}$ has a subsequence that converges uniformly on X to a continuous function f on X.

Note. The Arzelà-Ascoli Theorem is also covered in the Complex Analysis sequence (MATH 5510, MATH 5520). See my online notes for that sequence on Section VII.1. The Space of Continuous Functions $C(G, \Omega)$. The version of the Arzelà-Ascoli Theorem there is stated as:

Theorem VII.1.23. The Arzelà-Ascoli Theorem.

A set $\mathcal{F} \subset C(G, \Omega)$ is normal if and only if the following two conditions are satisfied:

(a) For each $z \in G$, we have that $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure, in Ω;

(b) $\mathcal{F}$ is equicontinuous at each point of G.

The set $C(G, \Omega)$ denotes the set of all continuous functions from open set $G \subseteq \mathbb{C}$ to metric space (Ω, d). A set $\mathcal{F} \subseteq C(G, \Omega)$ is *normal* if each sequence in $\mathcal{F}$ has a subsequence which converges to a function in f in $C(G, \Omega)$ (we called this "sequentially compact").

Note. The Arzelà-Ascoli Theorem can be generalized from real-valued functions on X to continuous functions mapping metric space X to metric space Y. Notice that we can define equicontinuity similar to above, but by replace the absolute value of the difference of function values to the metric distance between the function values in Y. With $C(X, Y)$ denoting the set of continuous function from X to Y, we get the following version of the result. This is part of Problem 10.8.

Theorem. Arzelà-Ascoli Theorem for $C(X, Y)$.

Let X be a compact metric space. Let $\{f_n\}$ be an equicontinuous sequence in $C(X, Y)$ such that for each $x \in X$, the closure of the set $\{f_n(x) \mid n \in \mathbb{N}\}$ is a compact subspace of Y. Then $\{f_n\}$ has a subsequence that converges uniformly on X to a function in $C(X, Y)$.

Note. Recall that the Heine-Borel Theorem implies that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. In a metric space in general, compactness implies closed and bounded (see problem 10.1). However, the converse does not hold as we saw in Example 9.5.A. Another example is given by the closed unit ball $\{f \in C[0,1] \mid \|f\|_{\max} \leq 1\}$ of $C[0,1]$. Of course it is closed and bounded, but it is not sequentially compact as seen by considering the sequence $f_n(x) = x^n$ which fails to have a subsequence that converges uniformly to a continuous function on $[0,1]$ (the pointwise limit exists, but equals a function $f(x)$ equal to 0 for $x \in [0,1)$ and equal to 1 at $x = 1$). So by the Characterization of Compactness for a Metric Space (Theorem 9.16; the contrapositive of (ii) implies (iii)), the closed unit ball is not compact. The Arzelà-Ascoli Theorem may be used to determine when a "in some sense closed and bounded" subset of $C(X)$ is compact, as follows.

Theorem 10.3. Let X be a compact metric space and $\mathcal{F}$ a subset of $C(X)$. Then $\mathcal{F}$ is a compact subspace of $C(X)$ if and only if $\mathcal{F}$ is closed, uniformly bounded, and equicontinuous.

Note. The compactness criterion of Theorem 10.3 for $C(X)$ has a counterpart in the setting of ℓ^p. For $1 \leq p < \infty$, a closed and bounded subset S of ℓ^p is compact if and only if it is equisummable in the sense that for each $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ for which

$$\sum_{k=N}^{\infty} |x_k|^p < \varepsilon \text{ for all } x = \{x_n\} \in S.$$

This is Problem 10.11.