Section 10.2. The Baire Category Theorem

Note. In this section we define "dense," "hollow," and "nowhere dense" sets in a metric space. The Baire Category Theorem concerns a countable union of open dense sets and a countable intersection of closed hollow sets. We address uniform boundedness of a family of continuous pointwise bounded real-valued functions on a complete metric space (in Theorem 10.6), and the continuity of a pointwise limit of a sequence of real-valued functions on a complete metric space (in Theorem 10.7).

Definition. Let E be a subset of a metric space X. A point $x \in E$ is an *interior point* of E provided there is an open ball centered at x that is contained in E. The set of interior points of E is the *interior* of E, denoted $\text{int}(E)$. A point $x \in X \sim E$ is an *exterior point* of E provided there is an open ball centered at x that is contained in $X \sim E$. The collection of exterior points of E is the *exterior* of E, denoted $\text{ext}(E)$. If a point $x \in X$ had the property that every ball centered at x contains points in E and points in $X \sim E$ then it is a *boundary point* of E. The collection of boundary points of E is the *boundary* of E and is denoted $\text{bd}(E)$.

Note. It is straightforward to show that for any subset E of X:

$$X = \text{int}(E) \cup \text{ext}(E) \cup \text{bd}(E) \text{ and the union is disjoint.} \qquad (5)$$

Note. Recall that a subset of A of a metric space X is *dense* (in X) if every nonempty open subset if X contains a point of A. The next definition, sort of goes in the opposite direction.

Definition. A subset of a metric space is *hollow* in X if its interior is empty.

Note 10.2.A. Notice that subset E of metric space X is hollow in X if and only if its complement, $X \sim E$, is dense in X.

Note 10.2.B. Consider a metric space X, $x \in X$, and $0 < r_1 < r_2$. By the continuity of the metric (see Note 9.3.A) we have the inclusion $\overline{B}(x, r_1) \subseteq B(x, r_2)$. We then have the inclusion $B(x, r_1) \subseteq \overline{B}(x, r_1) \subseteq B(x, r_2)$. Therefore, if $\mathcal{O}$ is an open subset of a metric space X, then for each point $x \in \mathcal{O}$ there is an open ball centered at x whose closure is contained in $\mathcal{O}$.

Theorem 10.2.A. The Baire Category Theorem.

Let X be a complete metric space.

(i) Let $\{\mathcal{O}_n\}_{n-1}^{\infty}$ be a countable collection of open dense subsets of X. Then the intersection $\cap_{n=1}^{\infty}\mathcal{O}_n$ is also dense.

(ii) Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of closed hollow subsets of X. Then the union $\cup_{n=1}^{\infty}F_n$ also is hollow.

Definition. A subset E of a metric space X is *nowhere dense* if its closure $\overline{E}$ is hollow.

Note 10.2.C. By Problem 10.16, a subset E of metric space X is nowhere dense if and only if for each open subset $\mathcal{O}$ of X, $E \cap \mathcal{O}$ is not dense in $\mathcal{O}$. Therefore, the Baire Category Theorem (Theorem 10.2.A) can be stated as: In a complete metric space the union of a countable collection of nowhere dense sets is hollow.

Corollary 10.4. Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X. If $\cup_{n=1}^{\infty} F_n$ has nonempty interior, then at least one of the F_n's has nonempty interior. In particular, if $X = \cup_{n=1}^{\infty} F_n$, then at least one of the F_n's has nonempty interior.

Corollary 10.5. Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X. Then $\cup_{n=1}^{\infty} \mathrm{bd}(F_n)$ is hollow.

Theorem 10.6. Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X that is pointwise bounded in the sense that for each $x \in X$, there is a constant M_x for which $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Then there is a nonempty open subset $\mathcal{O}$ of X on which $\mathcal{F}$ is uniformly bounded in the sense that there is a constant M for which $|f| \leq M$ on $\mathcal{O}$ for all $f \in \mathcal{F}$.

Note. We know that the limit of a pointwise convergent sequence of real-value continuous functions may not be continuous. However, if the convergence if uniform then the limit function must be continuous; see Theorem 8-2 of my online notes for Analysis 2 (MATH 4227/5227) on Section 8.1. Sequences of Functions. The next theorem addresses what happens in terms of a pointwise limit of a sequence of real-valued functions on a complete metric space. We'll see that the limit function must be continuous on a dense subset of the metric space.

Theorem 10.7. Let X be a complete metric space and $\{f_n\}$ a sequence of continuous real-valued functions on X that converges pointwise on X to the real-valued function f. Then there is a dense subset D of X for which $\{f_n\}$ is equicontinuous at each point in D.

Note. The following definitions are common terminology used in connection with the Baire Category Theorem (Theorem 10.2.A). Notice that we have not yet encountered the term "category" in this section!

Definition. A subset E of a metric space is of *first category* (or *meager*) if E is the union of a countable collection of nowhere dense subsets of X. A set that is not of first category is of *second category* (or *nonmeager*), and the complement of a set of first category is *residual* (or *co-meager*).

Note. In the terminology of categories, the Baire Category Theorem can be restated as:

The Baire Category Theorem (restated).
An open subset of a complete metric space is of the second category.

Note. René-Louis Baire (January 21, 1874–July 5, 1932) grew up in Paris at the time when the Eiffel tower was being constructed. He entered the École Normale Supérieure in 1891 and attended lectures at the Sorbonne by Hermite, Picard, and Pincar'e. During his oral examination he gave a poor performance in explaining the continuity of the exponential function. This would inspire some of his later research. He taught in a lycée (or high school) following his graduation and while there he worked on the theory of functions, the concept of limit, and limits of sequences of continuous functions. Baire was awarded a scholarship to study in Italy. While teaching at the lycée he wrote is dissertation on discontinuous functions. It was in his 1899 dissertation, *Sur les fonctions de variables réelles* ("On the Functions of Real Variables"), that he presented the Baire Category Theorem. Tired of teaching high school math, Baire took a job at the University of Montpellier in 1901. In 1904 Baire received a scholarship and spent a semester at the Collège de France where he lectured on the topics of dissertation. He published the lectures the following year. Baire suffered from poor health since he was young. Between 1909 and 1914 he continued teaching, but struggled with health issues. He took leave from teaching in 1914 to deal with his health and traveled to Lausanne, Switzerland. With the outbreak of the first world war, he spent 1914 to 1918 there. During his career, he carried on a contentious correspondence with Lebesgue and with de la

Vallée Poussin. After the war, Baire lived on the shores of Lake Geneva on the Switzerland/France boarder and receive some awards for his mathematics work. He retired in 1925 and spent his final years in relative solitude. He died by suicide in 1932. His major works are the book "Théorie des nombres irrationnels, des limites et de la continuité [Theory of Irrational Numbers, Limits, and Continuity]" published in 1905 and a two volume work "Leçons sur les théories générales de l'analyse [Lessons on the General Theory of Analysis]" published in 1907-08. These historical notes are largely based on the MacTutor History of Mathematics Archive biography of Baire, but includes some material on the Wikipedia page for Baire (accessed 12/16/2022).

René-Louis Baire (image from the MacTutor History of Mathematics Archive biography of Baire)

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