Section 10.3. The Banach Contraction Principle

Note. This is the one section in the whole text book that has the greatest potential to be considered as related to "applied math," in the traditional sense. Of course we could argue that Lebesgue measure and integration finds application in the L^p spaces (as in Chapters 7 and 8), or that this class finds applications to functional analysis and subsequently to probability theory... but that's not what most folks think about as "applications." We will mention applications of the Banach Contraction Principle (and some of its relatives) in numerical analysis and differential equations. We will prove the Picard Local Existence Theorem (Theorem 10.3.C), which gives conditions under which certain ordinary differential equations are guaranteed to have unique solutions.

Definition. Let X be a metric space and T a mapping of X to itself, $T : X \to X$. A point $x \in X$ is a *fixed point* of T if $T(x) = x$.

Note 10.3.A. In this section we find conditions that guarantee a function will have a fixed point. The result will involve limits and so require that the function be defined on a complete metric space. A real-valued function f of a real variable has a fixed point if and only if its graph $y = f(x)$ intersects the line $y = x$. In the special case where we have a continuous function $f : [a, b] \to \mathbb{R}$ for which $f([a, b]) \subseteq [a, b]$, a fixed point must exist. This follows from the Intermediate Value Theorem as applied to $g(x) = f(x) - x$ since g is continuous on $[a, b]$, $g(a) = f(a) - a \geq 0$, and $g(b) = f(b) - b \leq 0$, implying that $g(x_0) = f(x_0) - x_0 = 0$ for some $x_0 \in [a, b]$. In fact, you may have seen this result in Numerical Analysis (MATH 4257/5257). It appears (with the same proof) as Theorem 2.3(i); see my online notes for Numerical Analysis on Section 2.2. Fixed-Point Iteration. These Numerical Analysis notes include specific applications of the theorem.

Note 10.3.B. A subset K of $\mathbb{R}^n$ is *convex* if whenever $u, v \in K$, the line segment $\{tu + (1-t)v \mid 0 \leq t \leq 1\}$ joining u and v is contained in K (strictly speaking, the arithmetic statement "$tu + (1-t)v$" treats u and v as though they are vectors in $\mathbb{R}^n$). The following is an extension of Note 10.3.A from $\mathbb{R}$ (in which $[a, b]$ is a convex set) to the setting of $\mathbb{R}^n$:

Theorem 10.3.A. Brouwer's Fixed Point Theorem.
If K is a compact, convex subset of $\mathbb{R}^n$ and the mapping $T : K \to K$ is continuous, the T has a fixed point.

Royden and Fitzpatrick give as a reference for a proof of Brouwer's Fixed Point Theorem the book by Nelson Dunford and Jacob Schwartz, *Linear Operators, Part I*, John Wiley & Sons (1988); see Section V.10, "Fixed Point Theorems," for a statement of the result, and see Section V.12, "Notes and Remarks," for a proof (on pages 467–469). A proof of Brouwer's Fixed Point Theorem is given in Introduction to Algebraic Topology (not a formal ETSU class, but it should be the second class in a senior/graduate level sequence with Introduction to Topology [MATH 4357/5357] as the first part) in the special case that the compact, convex set is the closed unit disc in $\mathbb{R}^2$. See my online notes for Introduction to Algebraic Topology on Section 55. Retractions and Fixed Points (Theorem 55.6). Below, we will state and prove the Banach Contraction Principle which has a more restrictive condition on the mapping that does Brouwer's Fixed Point Theorem (namely, it requires the mapping to be a "contraction"), but only requires that the space under consideration to be complete (as opposed to the requirement of Brouwer's Fixed Point Theorem which requires that we deal with a compact, convex subset of $\mathbb{R}^n$).

Definition. A mapping T from a metric space (X, ρ) into itself is *Lipschitz* if there is a number $c \geq 0$, called the *Lipschitz constant* for the mapping, for which

$$\rho(T(u), T(v)) \leq c\, \rho(u, v) \text{ for all } u, v \in X.$$

If $c < 1$, the Lipschitz mapping is a *contraction*.

Note. For an introduction to Lipschitz functions, see my online notes for Complex Analysis 1 (MATH 5510) on Supplement. A Primer on Lipschitz Functions. In this supplement, properties of continuously differentiable, locally Lipschitz, continuity, Lipschitz, uniformly continuous, and continuous are compared for real-valued functions of a real variable (often on a compact set of real numbers).

Theorem 10.3.B. The Banach Contraction Principle.
Let X be a complete metric space and the mapping $T : X \to X$ be a contract. Then $T : X \to X$ has exactly one fixed point.

Note. Notice that the proof of the Banach Contraction Principle is constructive in nature. It claims the existence of a fixed point and then an algorithm is presented to that actually produces the fixed point (as a limit). Notice how this contrasts with the claim of the existence of a fixed point of function $f : [a, b] \to \mathbb{R}$ where $f([a, b]) \subseteq [a, b]$ in Note 10.3.A. In that example, the existence of a fixed point is established using the Intermediate Value Theorem, but there is no information on what the fixed point is (though in that case, the bisection method can be used to approximate the fixed point to any desired degree of accuracy; see my online notes for Numerical Analysis [MATH 4257/5257] on Section 2.1. The Bisection Method). In terms of numerical approximations, we see in the proof of the Banach Contraction Principle that, for initial point x_0, we have $\rho(x_m, x_k) \le \dfrac{c^k}{1-c}\rho(T(x_0), x_0)$ if $m > k$ (where x_n is the kth iterate of T applied to x_0, $x_n = T^n(x_k)$). In particular, with x_* as the (unknown) fixed point of T then we have (by continuity of the metric, see Note 9.3.A)

$$\rho(x_*, x_k) \le \frac{c^k}{1-c}\rho(T(x_0), x_0) \text{ for every } k \in \mathbb{N}.$$

Therefore, we can approximate x_* to any desired level of accuracy by making k sufficiently large so that the right hand side of the previous inequality is smaller than the desired level of accuracy.

Note. The Banach Contraction Principle is also sometimes called the "Contraction Mapping Theorem." The Contraction Mapping Theorem is also covered in Fundamentals of Functional Analysis (MATH 5740), though specific applications are not presented in that class. See my online notes on Section 2.12. Fixed Points and Contraction Mappings; notice Theorem 2.44. It may also be covered in Applied Math 1 (MATH 5610). See my online notes for Applied Math 1 (these are the version of the notes used in the fall 1996 class) on Section 3.3. The Contraction Mapping Theorem; notice Theorem 3.3.1. In this version of the Applied Math class, the Contraction Mapping Theorem is used to prove the existence of a solution to an initial value problem of the form $f'(x) = g(x, f(x))$, $f(x_0) = y_0$ where $g(x, y)$ is continuous and Lipschitz; see Theorem 3.4.2 in the notes on Section 3.4. The Initial Value Problem for One Scalar Differential Equation. We state and prove this result below (see the Picard Local Existence Theorem).

Note 10.3.C. We now briefly discuss solutions of (nonlinear) differential equations. Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0). We consider the problem of, given function $g : \mathcal{O} \to \mathbb{R}$, finding an open interval of real numbers I containing x_0 and a differentiable function $f : I \to \mathbb{R}$ such that

$$\begin{aligned} f'(x) = &\ g(x, f(x)) \text{ for all } x \in I \\ f(x_0) = &\ y_0. \end{aligned} \tag{14}$$

A special case is when g is independent of its second variable and $g(x, y) = h(x)$. In the event that h is continuous, we have by the Fundamental Theorem of Calculus, Part 1 (see my online notes for Calculus 1 [MATH 1910] on Section 5.4. The Fundamental Theorem of Calculus; notice Theorem 5.4(a)) that the unique solution of (14) is given by

$$f(x) = y_0 + \int_{x_0}^{x} h(t)\, dt \text{ for all } x \in I.$$

Notice that this is a "symbolic" solution; to find a more useful solution requires us to find an antiderivative of h on I. You see the difficulty of this in Calculus 2 (MATH 1920) when you learn techniques of integration. If h is analytic on I (a condition much more restrictive than continuity), then it is easy to find a power series representation for a solution. For a general continuous real-valued function of two variables g (that is, g is *not* independent of the second variable), if function $f : I \to \mathbb{R}$ is continuous and $(x, f(x)) \in \mathcal{O}$ for each $x \in I$, then f is a solution of the differential equation (14) if and only if (again, by the Fundamental Theorem of Calculus)

$$f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \text{ for all } x \in I. \tag{15}$$

Equation (15) is an *integral equation* in unknown function f. We use this representation of differential equation (14) in the proof of the Picard Local Existence Theorem, which gives the existence ("locally") of a unique solution of the DE (14).

Theorem 10.3.C. The Picard Local Existence Theorem.

Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ containing the point (x_0, y_0). Suppose the function $g : \mathcal{O} \to \mathbb{R}^2$ is continuous and there is a positive number M for which the following Lipschitz property in the second variable holds uniformly with respect to the first variable:

$$|g(x, y_1) - g(x, y_2)| \le M|y_1 - y_2| \text{ for all points } (x, y_1) \text{ and } (x, y_2) \text{ in } \mathcal{O}. \tag{16}$$

Then there is an open interval I containing x_0 on which the following differential equation has a unique solution:

$$\begin{aligned} f'(x) &= g(x, f(x)) \text{ for all } x \in I \\ f(x_0) &= y_0. \end{aligned} \tag{14}$$