Chapter 11. Topological Spaces: General Properties

Section 11.1. Open Sets, Closed Sets, Bases, and Subbases

Note. In this section, we define a topological space. In this setting, since we have a collection of open sets, we can still accomplish a study of the standard topics of analysis (such as limits, continuity, convergence, and compactness).

Definition. Let X be a nonempty set. A topology $\mathcal{T}$ on X is a collection of subsets of X, called *open sets*, possessing the following properties:

(i) The entire set X and the empty-set $\varnothing$ are open.

(ii) The intersection of any finite collection of open sets is open.

(iii) The union of any collection of open sets is open.

A nonempty set X, together with a topology on X, is called a *topological space*, denoted $(X, \mathcal{T})$. For a point $x \in X$, an open set that contains x is called a *neighborhood* of x.

Note. Of course, with $X = \mathbb{R}$ and $\mathcal{T}$ equal to the set of all "traditionally" open sets of real numbers, we get the topological space $(\mathbb{R}, \mathcal{T})$. $\mathcal{T}$ is called the *usual topology* on $\mathbb{R}$. Another topology on $\mathbb{R}$ is given by taking $\mathcal{T}$ equal to the power set of $\mathbb{R}$, $\mathcal{T} = \mathcal{P}(\mathbb{R})$.

Proposition 11.1. A subset E of a topological space X is open if and only if for each point $x \in E$ there is a neighborhood of x that is contained in E.

Example. If you are familiar with metric spaces then for (X, ρ) a metric space, define $\mathcal{O} \subset X$ to be open if for each $x \in \mathcal{O}$ there is an open ball $\{y \in X \mid \rho(x, y) < \varepsilon\}$ centered at x contained in $\mathcal{O}$. This is the *metric topology* induced by metric ρ.

Example. Let X be a nonempty set. Define $\mathcal{T}$ to be the power set of X, $\mathcal{T} = \mathcal{P}(X)$. Then $\mathcal{T}$ is a topology on X called the *discrete topology*.

Example. Let X be a nonempty set. Define $\mathcal{T} = \{\varnothing, X\}$. Then $\mathcal{T}$ is a topology on X called the *trivial topology* on X.

Example. Let $(X, \mathcal{T})$ be a topological space and let $E \subset X$. Define $\mathcal{S} = \{E \cap \mathcal{O} \mid \mathcal{O} \in \mathcal{T}\}$. Then $(E, \mathcal{S})$ is a topological space called a *subspace* of $(X, \mathcal{T})$.

Note. We know that every open set of real numbers (under the usual topology) is a countable disjoint union of open intervals. So every open set of real numbers is "made up of" intervals. This is the idea behind the following definition.

Definition. For a topological space $(X, \mathcal{T})$ and a point $x \in X$, a collection of neighborhoods of x, $\mathcal{B}_x$, is a *base for the topology at* x if for any neighborhood $\mathcal{U}$ of x in $\mathcal{T}$ there is a set $B \in \mathcal{B}_x$ for which $\mathcal{B} \subset \mathcal{U}$. A collection of open sets $\mathcal{B}$ is a *base for the topology* $\mathcal{T}$ if it contains a base for the topology at each point.

Note. The set of all open intervals is a base for the usual topology on $\mathbb{R}$. The set of all intervals with rational endpoints is a countable base for the usual topology on $\mathbb{R}$. The following result classifies a collection of subsets of X as a base.

Proposition 11.2. For a nonempty set X, let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a base for a topology for X if and only if

(i) $\mathcal{B}$ covers X (that is, $X = \cup_{B \in \mathcal{B}} B$).

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a set $B \in \mathcal{B}$ for which $x \in B \subset B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\varnothing$ and unions of subcollections of $\mathcal{B}$.

Example. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces. Consider the set $X \times Y$. Define $\mathcal{B} = \{\mathcal{O}_1 \times \mathcal{O}_2 \mid \mathcal{O}_1 \in X, \mathcal{O}_2 \in Y\}$. Then we claim that $\mathcal{B}$ is a base for topology on $X \times Y$, called the *product topology*.

Definition. For a topological space $(X, \mathcal{T})$, a subcollection $\mathcal{S}$ of $\mathcal{T}$ that covers X is a *subbase* for the topology $\mathcal{T}$ provided intersections of finite subcollections of $\mathcal{S}$ are a base for $\mathcal{T}$.

Note. We have already seen that the open intervals are a base for the usual topology on $\mathbb{R}$. Since finite intersections of open intervals yield open intervals or $\varnothing$, then the collection of open intervals is also a subbase for the usual topology on $\mathbb{R}$. The same can be said of the collection of open intervals with rational endpoints.

Note. Now we give several "topological definitions," each of which you have encountered before (in metric spaces, say). However, we do not need a metric, only the presence of open sets. Notice that the idea of "closeness," even in the absence of a metric!

Definition. For a subset E of a topological space $(X, \mathcal{T})$, a point $x \in X$ is a *point of closure* of E provided every neighborhood of x contains a point in E. The collection of closure points of E is the closure of E, denoted $\overline{E}$. If $E = \overline{E}$ then set E is *closed*.

Proposition 11.3. For E a subset of a topological space $(X, \mathcal{T})$, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Definition. Some other topological ideas are defined in Problem 11.5.

(i) A point $x \in X$ is an *interior point* of set E provided there is a neighborhood of x that is contained in E. The collection of all interior points of set E is the *interior* of E, denoted $\text{int}(E)$.

(ii) A point $x \in X$ is an *exterior point* of set E provided there is a neighborhood of x that is contained in $X \sim E$. The collection of all exterior points of E is the *exterior* of E, denoted $\text{ext}(E)$.

(iii) A point $x \in X$ is a *boundary point* of E provided every neighborhood of x contains points in E and points in $X \sim E$. The collection of all boundary points of E is the *boundary* of E, denoted $\text{bd}(E)$ or $\partial(E)$.

Note. In Problem 11.5, it is shown that:

(i) $\text{int}(E)$ is open and set E is open if and only if $E = \text{int}(E)$.

(ii) $\text{ext}(E)$ is open and set E is open if and only if $\overline{E} \sim E \subseteq \text{ext}(E)$.

(iii) $\partial(E)$ is closed and set E is open if and only if $E \cap \partial(E) = \varnothing$. Set E is closed if and only if $\partial(E) \subseteq E$.

Note. Recall that *we* have defined a set E to be closed if $E = \overline{E}$. The following result shows that our use of "closed" is consistent here (in topological spaces) with its use elsewhere.

Proposition 11.4. A subset of a topological space $(X, \mathcal{T})$ is open if and only if its complement in X is closed.

Note. Of course, a set E is closed if and only if its complement is open. This fact, combined with DeMorgan's Laws, give the following.

Proposition 11.5. Let $(X, \mathcal{T})$ be a topological space.

(i) $\varnothing$ and X are closed.

(ii) The union of any finite collection of closed subsets of X is closed.

(iii) The intersection of any collection of closed subsets of X is closed.

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