Section 11.2. The Separation Properties

Note. I like the way Royden and Fitzpatrick start this section: "In order to establish interesting results for topological spaces and continuous mapping between such spaces, it is necessary to enrich the rudimentary topological structure. In this section we consider so-called separation properties for a topology on a set X, which ensure that the topology discriminates between certain disjoint pairs of sets and, as a consequence, ensure that there is a subset collection of continuous real-valued functions on X."

Definition. Let $K \subseteq X$ in a topological space $(X, \mathcal{T})$. A *neighborhood of set* K is some open $\mathcal{O} \in \mathcal{T}$ where $K \subseteq \mathcal{O}$. Two disjoint subsets A, B of X can be *separated by disjoint neighborhoods* if there exist $\mathcal{O}_A, \mathcal{O}_B \in \mathcal{T}$ such that $A \subset \mathcal{O}_A$, $B \subset \mathcal{O}_B$, and $\mathcal{O}_A \cap \mathcal{O}_B = \varnothing$.

Definition. We consider various topological spaces $(X, \mathcal{T})$ satisfying separation properties of the following types:

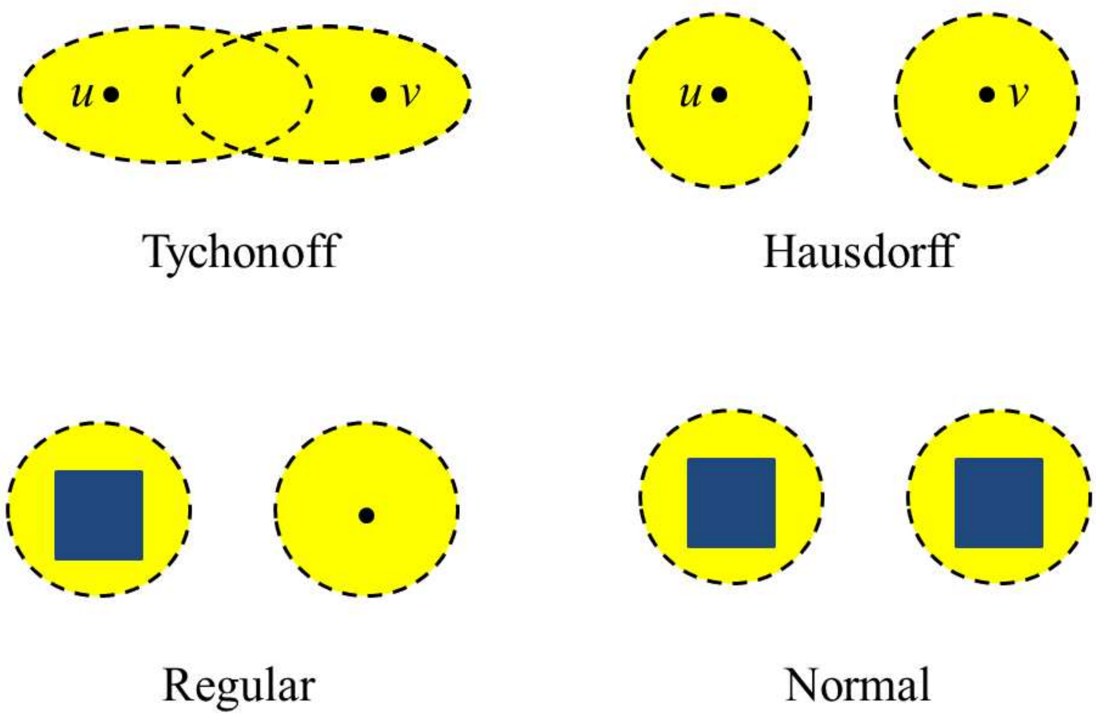
The Tychonoff Separation Property. For each two points $u, v \in X$, there is a neighborhood of u that does not contain v and there is a neighborhood of v that does not contain u.

The Hausdorff Separation Property. Each two points in X can be separated by disjoint neighborhoods.

The Regular Separation Property. The Tychonoff separation property holds and, moreover, each closed set and point not in the set can be separated by disjoint neighborhoods.

The Normal Separation Property. The Tychonoff separation property holds and, moreover, each two disjoint closed sets can be separated by disjoint neighborhoods.

Note. To illustrate the separation properties consider the following pictures.



Note. We'll see relationships between spaces satisfying these separation properties soon. The following result classifies Tychonoff spaces.

Proposition 11.6. A topological space $(X, \mathcal{T})$ is a Tychonoff space if and only if every set consisting of a single point is closed.

Proposition 11.7. Every metric space is normal.

Note. Based on Proposition 11.7, every topological space where the topology is based on a metric is a normal space. By Proposition 11.6, in a Tychonoff space every singleton $\{x\}$ forms a closed set, so all normal spaces are also regular. If a space is Hausdorff, C is a closed set in the space, and x is a point in the space, then

$$\mathcal{O} = \sup_{y \in C}\{\mathcal{O}_y \mid \mathcal{O}_y \text{ is open}, y \in \mathcal{O}_y, x \notin \mathcal{O}_y\}$$

is open, $C \subseteq \mathcal{O}$, and $x \notin \mathcal{O}$ and the space is regular. Of course, any Hausdorff space is Tychonoff. So we have (schematically):

$$\mathcal{T}_{\text{metric}} \subseteq \mathcal{T}_{\text{normal}} \subseteq \mathcal{T}_{\text{regular}} \subseteq \mathcal{T}_{\text{Hausdorff}} \subseteq \mathcal{T}_{\text{Tychonoff}}.$$

Note. The following classifies normal spaces in terms of the behavior of nested closed sets.

Proposition 11.8. Let $(X, \mathcal{T})$ be a Tychonoff topological space. Then X is normal if and only if whenever $\mathcal{U}$ is a neighborhood of a closed subset F of X, then there is another neighborhood of F whose closure is contained in $\mathcal{U}$; that is, there is an open $\mathcal{O}$ for which $F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$.