Section 11.3. Countability and Separability

Note. In this brief section, we introduce sequences and their limits. This is applied to topological spaces with certain types of bases.

Definition. A sequence $\{x_n\}$ in a topological space X *converges* to a point $x \in X$ provided for each neighborhood $\mathcal{U}$ of x, there is an index N such that $n \geq N$, then x_n belongs to $\mathcal{U}$. The point x is a *limit* of the sequence.

Note. We can now elaborate on some of the results you see early in a senior level real analysis class. You prove, for example, that the limit of a sequence of real numbers (under the usual topology) is unique. In a topological space, this may not be the case. For example, under the trivial topology every sequence converges to every point (at the other extreme, under the discrete topology, the only convergent sequences are those which are eventually constant). For an additional example, see Bonus 1 in my notes for Analysis 1 (MATH 4217/5217) at: `http://faculty.etsu.edu/gardnerr/4217/notes/3-1.pdf`. In a Hausdorff space (as in a metric space), points can be separated and limits of sequences are unique.

Definition. A topological space $(X, \mathcal{T})$ is *first countable* provided there is a countable base at each point. The space $(X, \mathcal{T})$ is *second countable* provided there is a countable base for the topology.

Example. Every metric space (X, ρ) is first countable since for all $x \in X$, the countable collection of open balls $\{B(x, 1/n)\}_{n=1}^{\infty}$ is a base at x for the topology induced by the metric.

Example. The base of all intervals of real numbers with rational endpoints for the usual topology shows that the usual topology on $\mathbb{R}$ is second countable.

Proposition 11.9. Let $(X, \mathcal{T})$ be a first countable topological space. For a subset E of X, a point $x \in X$ is a point of closure of E if and only if x is a limit point of a sequence in E. Therefore, a subset E of X is closed if and only if whenever a sequence in E converges to $x \in X$, the point x belongs to E.

Note. The hypothesis of first countable is necessary in Proposition 11.9. In Problem 11.22(ii) you are asked to show that there is a point of closure of a set that is not a limit of a sequence in the set—the space is not first countable.

Definition. A subset E of topological space $(X, \mathcal{T})$ is *dense* in X provided every open set in $\mathcal{T}$ contains a point of E. $(X, \mathcal{T})$ is *separable* it it has a countable dense subset.

Note. If E is dense then $\overline{E} = X$. Of course, $\mathbb{Q}$ is dense in $\mathbb{R}$ under the usual topology (and so $\mathbb{R}$ is separable).

Note. In Proposition 9.25, it is shown that a metric space is second countable if and only if it is separable. In a general topological space, second countable implies separable (Problem 11.19) but a separable space (even one which is first countable) may fail to be second countable (Problem 11.21).

Definition. A topological space is *metrizable* provided the topology is induced by a metric.

Note. Since every metric space is normal (Proposition 11.7), then we see that any non-normal topological space is not metrizable. For example, any topological space under the trivial topology is not metrizable. We desire to classify metrizable spaces. The following result classifies second countable topological spaces. The proof is given in section 12.1 (see page 242).

Theorem. The Urysohn Metrization Theorem.
Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is metrizable if and only if it is normal.

Note. Royden and Fitzpatrick mention the Nagata-Smirnov-Bing Metrization Theorem which classifies metrizable topological spaces. We now briefly explain this result by quoting John Kelley's *General Topology* (Van Nostrand Co., 1955). You can find a PDF of this book at archive.org at (accessed 5/1/2015):

`https://ia700608.us.archive.org/23/items/`
`GeneralTopology/Kelley-GeneralTopology.pdf`.

Definition. (Kelley, pages 126 and 127.) A family $\mathcal{A}$ of subsets of a topological space is *locally finite* provided each point of the space has a neighborhood which intersects only finitely many members of $\mathcal{A}$. A family $\mathcal{A}$ is *discrete* if each point of the space has a neighborhood which intersects at most one member of $\mathcal{A}$. A family $\mathcal{A}$ is *σ-locally finite* if and only if it is the union of a countable number of locally finite subfamilies. A family $\mathcal{A}$ is *σ-discrete* if and only if it is the union of a countable number of discrete subfamilies.

Theorem. The Metrization Theorem. (Kelley, page 127.)
The following three condition on a topological space are equivalent.

(a) The space is metrizable.

(b) The space is regular and the topology has a σ-locally finite base.

(c) The space is regular and the topology has a σ-discrete base.

Note. In Kelley's book, he states parts (b) and (c) of the Metrization Theorem as requiring the space to be "T_1 and regular." A space is T_1 if singletons form closed sets. Royden and Fitzpatrick define a "regular space" to be one which also satisfies the Tychonoff separation property, and we know by Proposition 11.6 that every singleton forms a closed set in a Tychonoff space. So Royden and Fitzpatrick's "regular" is equivalent to Kelley's "T_1 and regular." So we have stated the Metrization Theorem using Royden and Fitzpatrick's terminology.

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