Section 11.4. Continuous Mappings Between Topological Spaces

Note. In this section, we deal with continuous functions and some of their properties.

Note. The following definition is a parallel to the standard ε-δ definition of continuity where the "for all $\varepsilon > 0$" has been replaced with "for any neighborhood" and "there exists $\delta > 0$" has been replaced with "there exists a neighborhood."

Definition. For topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$, a mapping $f : X \to Y$ is *continuous at a point* $x_0 \in X$ provided that for any neighborhood $\mathcal{O}$ of $f(x_0)$, there exists a neighborhood $\mathcal{U}$ of x_0 for which $f(\mathcal{U}) \subseteq \mathcal{O}$. f is *continuous* if it is continuous at each point in X.

Proposition 11.10. A mapping $f : X \to Y$ between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is continuous if and only if for any subset $\mathcal{O} \in \mathcal{S}$, its inverse image under f, $f^{-1}(\mathcal{O}) \in \mathcal{T}$.

Proposition 11.11. The composition of continuous mappings between topological spaces when defined, is continuous.

Note. The straightforward proof of Proposition 11.11 is Problem 11.26.

Definition. Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then $\mathcal{T}_2$ is *weaker* than $\mathcal{T}_1$ and $\mathcal{T}_1$ is *stronger* than $\mathcal{T}_2$.

Note. The more open sets a topology has, the "harder" it is for sequences to converge (for example)—thus the use of "stronger" and "weaker."

Proposition 11.12. Let X be a nonempty set and $\mathcal{S}$ any collection of subsets of X that covers X. The collection of subsets of X consisting of intersections of finite subcollections of $\mathcal{S}$ is a base for a topology $\mathcal{T}$ for X. It is the weakest topology containing $\mathcal{S}$ in the sense that if $\mathcal{T}'$ is any other topology for X containing $\mathcal{S}$ then $\mathcal{T} \subseteq \mathcal{T}'$.

Note. The proof of Proposition 11.12 is left as Problem 11.27.

Definition. Let X be a nonempty set and consider a collection of mappings $\mathcal{F} = \{f_\alpha : X \to X_\alpha\}_{\alpha \in \Lambda}$, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\mathcal{F} = \{f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F}, \mathcal{O}_\alpha \text{ open in } X_\alpha\}$$

is the *weak topology* for X induced by $\mathcal{F}$.

Note. The following result more clearly elaborates on the meaning of the weak topology.

Proposition 11.13. Let X be a nonempty set and $\mathcal{F} = \{f_\lambda : X \to X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each X_λ is a topological space. The weak topology for X induced by $\mathcal{F}$ is the topology on X that has the fewest number of sets among the topologies on X for which each mapping $f_\lambda : X \to X_\lambda$ is continuous.

Definition. A continuous mapping from a topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{S})$ is a *homeomorphism* provided it is one to one, maps X onto Y, and has continuous inverse f^{-1} from Y to X. If there is a homeomorphism between two topological spaces, the spaces are *homeomorphic*.

Note. Homeomorphic topological spaces are indistinguishable from each other *as topological spaces*. So if $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is a homeomorphism then $E \subseteq X$ is open in $(X, \mathcal{T})$ if and only if $f(E)$ is open in $\mathcal{S}$. However, this does not mean that the two spaces share other properties. In the next example, we show that $L^1(E)$ is homeomorphic to $L^2(E)$ for measurable set E.

Example. Mazur's Example.

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable. For $f \in L^1(E)$, define function $\Phi(f)$ on E by $\Phi(f)(x) = \text{sgn}(f(x))|f(x)|^{1/2}$. Then $\Phi(f)$ clearly belongs to $L^2(E)$. We "leave as an exercise" (a tedious one) to show that for any $a, b \in \mathbb{R}$ we have

$$|\text{sgn}(a)|a|^{1/2} - \text{sgn}(b)|b|^{1/2}| \leq 2|a - b|.$$

If follows from this that $\|\Phi(f) - \Phi(g)\|_2^2 \leq 2\|f - g\|_1$ for all $f, g \in L^1(E)$. So Φ is a continuous mapping (to make $\|\Phi(f) - \Phi(g)\|_2 < \varepsilon$, consider $f, g \in L^1(E)$ such that

$\|f - g\|_1 < \delta = \varepsilon^2/2$). In fact, we see that Φ is uniformly continuous on $L^1(E)$. Also Φ is one to one since $\Phi(f) = \Phi(g)$ implies that $f = g$. Now Φ is onto $L^2(E)$ since its inverse is $\Phi^{-1}(f)(x) = \mathrm{sgn}(f(x))|f(x)|^2$ for $f \in L^2(E)$:

$$\Phi(\mathrm{sgn}(f(x))|f(x)|^2) = \mathrm{sgn}(\mathrm{sgn}(f(x))|f(x)|)|\mathrm{sgn}(f(x))|f(x)|^2|^{1/2}$$

$$= \mathrm{sgn}(f(x))|f(x)| = f(x).$$

Problem 11.38 gives that for all $a, b \in \mathbb{R}$, $|\mathrm{sgn}(a)|a|^2 - \mathrm{sgn}(b)|b|^2| \leq 2|a-b|(|a|+|b|)$. Therefore $\|\Phi^{-1}(f) - \Phi^{-1}(g)\|_1 \leq 2\|f-g\|_2^2(\|f\|_2^2 + \|g\|_2^2)$. So $\Phi^{-1} : L^2(E) \to L^1(E)$ is continuous: For given $f \in L^2(E)$, to make $\Phi^{-1}(f) - \Phi^{-1}(g)\|_1 < \varepsilon$, consider $g \in L^2(E)$ such that $\|f - g\|_2 < \delta$ where $\delta = \min\{\|f\|_2, \sqrt{\varepsilon}/(\sqrt{10}\|f\|_2)\}$. We have for $\|f - g\|_2 < \delta$ that $\|g\|_2 < \|f\|_2 + \delta < 2\|f\|_2$ and so

$$\|\Phi^{-1}(f) - \Phi^{-1}(g)\|_1 \leq 2\|f-g\|_2^2(\|f\|_2^2 + \|g\|_2^2) < 2\|f-g\|_2^2(\|f_2^2 + (2\|f\|_2)^2)$$

$$= 10\|f-g\|_2^2\|f\|_2^2 < 10\left(\frac{\sqrt{\varepsilon}}{\sqrt{10}\|f\|_2}\right)^2 \|f\|_2^2 = \varepsilon.$$

Therefore $\Phi : L^1(E) \to L^2(E)$ is a homeomorphism and $L^1(E)$ is (topologically) isomorphic to $L^2(E)$. Of course, as Banach spaces, $L^1(E)$ and $L^2(E)$ are quite different.

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