Section 11.5. Compact Topological Spaces

Note. Since compactness is defined in terms of open sets, then it is a topological property. However, since a topological space may not be metrizable, we will not see a version of the Heine-Borel Theorem here ("boundedness" may have no meaning in a general topological space).

Definition. A collection of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ is a *cover* of set E if $E \subseteq \cup_{\lambda\in\Lambda} E_\lambda$. If in addition each E_λ is open then $\{E_\lambda\}_{\lambda\in\Lambda}$ is an *open cover* of E. A subset of $\{E_\lambda\}_{\lambda\in\Lambda}$ is called a *subcover* of $\{E_\lambda\}_{\lambda\in\Lambda}$.

Definition. A topological space $(X, \mathcal{T})$ is *compact* provided that every open cover of X has a finite subcover. A subset K of X is *compact* provided K, considered as a topological space with the subspace topology form X, is compact.

Definition. A collection of sets in a topological space has the *finite intersection property* provided every finite subcollection has nonempty intersection.

Note. The following two propositions hold in both metric spaces and topological spaces.

Proposition 11.14. A topological space $(X, \mathcal{T})$ is compact if and only if every collection of closed subsets of X that possesses the finite intersection property has nonempty intersection.

Proposition 11.15. A closed subset of a compact topological space $(X, \mathcal{T})$ is compact.

Note. We now present results similar to results seen in the metric space setting, but slightly different because they require specific properties of a topological space (such as Hausdorff).

Proposition 11.16. A compact subspace K of a Hausdorff topological space $(X, \mathcal{T})$ is a closed subset of K.

Definition. A topological space $(X, \mathcal{T})$ is said to be *sequentially compact* provided each sequence in $(X, \mathcal{T})$ has a subsequence that converges to a point of X.

Note. A metric space is compact if and only if it is compact (see Theorem 9.16). The same holds for a topological space when it is second countable, as the following shows.

Proposition 11.17. Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is compact if and only if it is sequentially compact.

Theorem 11.18. A compact Hausdorff space is normal.

Note. We now apply a continuous function to a topological space. The following is familiar from senior-level analysis.

Proposition 11.20. The continuous image of a compact topological space is compact.

Note. The following is our version of the Extreme Value Theorem.

Corollary 11.21. A continuous real-valued function on a compact topological space takes a maximum and minimum functional value.

Proposition 11.19. A continuous one to one mapping f of a compact space $(X, \mathcal{T})$ onto a Hausdorff space Y is a homeomorphism.

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