Section 11.6. Connected Topological Spaces

Note. In this section, we define what it means to "separate" a topological space. A space without a separation is then said to be connected. This approach is very similar to that taken in senior level Analysis 1 (MATH 4217/5217) when defining a connected set of real numbers. You will recall that a set of real numbers (under the usual topology) is connected if and only if it is either an interval or a singleton. For more details, see my online notes: `http://faculty.etsu.edu/gardnerr/` `4217/notes/3-1.pdf` (Theorem 3-14). Connected sets in other spaces can be more complicated. For example, in $\mathbb{C}$ (and $\mathbb{R}^2$ treated as a metric space) a set can be connected but not "simply connected."

Note. On page 237, Royden and Fitzpatrick comment that a set of real numbers is connected if and only if it is an interval. This contradicts the classification of connected sets of real numbers as *either* intervals *or* singletons. This is resolved by Royden and Fitzpatrick by calling an "interval" of the form $[a, b]$, where $a = b$, a *degenerate interval* (see page 109).

Recall. Two nonempty open subsets $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{R}$ *separate* set $X \subset \mathbb{R}$ if they are disjoint and their union is X (that is, $\mathcal{U} \cap \mathcal{V} = \varnothing$ and $X = \mathcal{U} \cup \mathcal{V}$). Set $X \subset \mathbb{R}$ which cannot be separated by such a pair is *connected*. We have a slightly different definition in a topological space.

Note. Since $\tilde{\mathcal{U}} = X \sim \mathcal{V}$ and $\tilde{\mathcal{V}} = X \sim \mathcal{U}$, then for $\mathcal{U}$ and $\mathcal{V}$ a separation of X, $\mathcal{U}$ and $\mathcal{V}$ are both open and closed. So a topological space is connected if and only if the only sets which are both open and closed are X and $\varnothing$.

Definition. In topological space $(X, \mathcal{T})$, set E is *connected* if there do not exist $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$ such that

(i) $\mathcal{O}_1 \cap E \neq \varnothing$, $\mathcal{O}_2 \cap E \neq \varnothing$,

(ii) $E \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$, and

(iii) $E \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$.

Note. In all this ε-δ, "inverse image of open sets are open" stuff which you have seen in connection with continuous functions over the past few years, you may have lost touch with the key concept behind a continuous function: It doesn't break things into pieces! This idea is revived in the following result.

Proposition 11.22. Let f be a continuous mapping of a connected space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{S})$. Then its image $f(X)$ is connected.

Note. The following definition allows us to bring our old friend from Calculus 1, the Intermediate Value Theorem, into the setting of connected topological spaces.

Definition. A topological space X is said to have the *intermediate value property* provided the image of any continuous real valued function on X is an interval.

Proposition 11.23. A topological space has the intermediate value property if and only if it is connected.

Note. Since intervals in $\mathbb{R}$ are connected sets, then the image of an interval under a continuous function is connected. So a topological space $(X, \mathcal{T})$ is connected *if* for each pair of points $u, v \in X$, there is a continuous map $f : [0, 1] \to X$ for which $f(0) = u$ and $f(1) = v$. A topological space possessing this type of connectivity is called *arcwise connected* (or *path connected*). Every arcwise connected space is connected, but a space may be connected and fail to be arcwise connected, as shown in Problem 11.49.

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