Section 12.2. The Tychonoff Product Theorem

Note. In this section, we introduce the Cartesian product of topological spaces. We will see such desirable properties as that a product of Tychonoff spaces is Tychonoff (Problem 12.11), a product of Hausdorff spaces is Hausdorff (Problem 12.12), and a product of a countable number of sequentially compact topological spaces is sequentially compact (Problem 12.21). The Tychonoff Product Theorem itself claims that a product of compact topological spaces is compact.

Definition. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of sets indexed by set Λ. The *Cartesian product* $\prod_{\lambda\in\Lambda} X_\lambda$ is the collection of mappings from the index set Λ to the union $\cup_{\lambda\in\Lambda} X_\lambda$ such that each index $\lambda \in \Lambda$ is mapped to a member of X_λ. For $x \in \prod_{\lambda\in\Lambda} X_\lambda$, denote $x(\lambda)$ as x_λ, the λ-th *component* of x. For each $\lambda_0 \in \Lambda$ define the λ_0 *projection* mapping $\pi_{\lambda_0} : \prod_{\lambda\in\Lambda} X_\lambda \to X_{\lambda_0}$ as $\pi_{\lambda_0}(x) = x_{\lambda_0}$ for $x \in \prod_{\lambda\in\Lambda} X_\lambda$.

Note. We want to put a topology on a Cartesian product of topological spaces. For a finite Cartesian product (that is, for a finite index set) we just define the topology as the product of all $\mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_n$ where set $\mathcal{O}_k$ is open in space $(X_k, \mathcal{T}_k)$. We do a similar thing for arbitrary index set Λ.

Definition. Let $\{(X_\lambda, \mathcal{T}_\lambda)\}_{\lambda\in\Lambda}$ be a collection of topological spaces indexed by arbitrary set Λ. The *product topology* on the Cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ is the topology that has as a base sets of the form $\prod_{\lambda\in\Lambda} \mathcal{O}_\lambda$ where each $\mathcal{O}_\lambda \in \mathcal{T}_\lambda$ is open and $\mathcal{O}_\lambda = X_\lambda$ except for finitely many λ.

Note. If all the X_λ's are the same, we denote the Cartesian product as X^Λ.

Proposition 12.3. Let X be a topological space. A sequence $\{f_n : \Lambda \to X\}$ converges to f in the product space X^Λ if and only if $\{f_n(\lambda)\}$ converges to $f(\lambda)$ for each $\lambda \in \Lambda$. Thus, convergence of a sequence with respect to the product topology is pointwise convergence.

Proposition 12.4. The product topology on the Cartesian product of topological spaces $\prod_{\lambda\in\Lambda} X_\lambda$ is the weak topology associated to the collection of projections $\{\pi_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda\}_{\lambda\in\Lambda}$, that is, it is the topology on the Cartesian product that has the fewest number of sets among the topologies for which all the projection mappings are continuous.

Definition/Recall. A collection $\mathcal{F}$ of sets in set X has the *finite intersection property* if any finite subcollection of $\mathcal{F}$ has a nonempty intersection.

Note. We now prove two lemmas which are needed in the proof of the Tychonoff Product Theorem, which states that products of compact spaces are compact.

Lemma 12.5. Let $\mathcal{A}$ be a collection of subsets of a set X that possesses the finite intersection property. Then there is a collection $\mathcal{B}$ of subsets of X which contains $\mathcal{A}$, has the finite intersection property, and is maximal with respect to this property; that is, no collection of subsets of X that properly contains $\mathcal{B}$ possesses the finite intersection property.

Note. The proof of Lemma 12.5 uses Zorn's Lemma and so gives us another result (The Tychonoff Product Theorem) which is implied by the Axiom of Choice (or more appropriately, its equivalent Zorn's Lemma).

Lemma 12.6. Let $\mathcal{B}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then each intersection of a finite property. Then each intersection of a finite number of sets in $\mathcal{B}$ is again in $\mathcal{B}$, and each subset of X that has nonempty intersection with each set in $\mathcal{B}$ is itself in $\mathcal{B}$.

The Tychonoff Product Theorem.

Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a collection of compact topological spaces indexed by a set Λ. Then the Cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$, with the product topology, also is compact.

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