Section 12.3. The Stone-Weierstrass Theorem

Note. In this section, we state and prove a result concerning continuous real-valued functions on a compact Hausdorff space. Royden and Fitzpatrick motivate this result by stating "one of the jewels of classical analysis:"

The Weierstrass Approximation Theorem.
Let f be a continuous real-valued function on a closed, bounded interval $[a, b]$. Then for each $\varepsilon > 0$, there is a polynomial p for which $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$.

Note. Anton R. Schep of the University of South Carolina has a nice, concise and self-contained proof of the Weierstrass Approximation Theorem posted online at:

`http://people.math.sc.edu/schep/weierstrass.pdf`

Dr. Schep's proof is essentially the proof of Weierstrass, which appeared originally in "Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen," *Sitzungsberichte der Kniglich Preuischen Akademie der Wissenschaften zu Berlin*, 1885 (11).

Definition. For compact Hausdorff space X, define the linear space $C(X)$ of continuous real-valued functions on X with the maximum norm $\|f\| = \max_{x \in X} |f(x)|$.

Note. The Weierstrass Approximation Theorem implies that the polynomials are dense in $C([a, b])$ (as a normed linear space).

Definition. A linear subspace $\mathcal{A}$ of $C(X)$ is an *algebra* if the product of any two functions in $\mathcal{A}$ belongs to $\mathcal{A}$. A collection $\mathcal{A}$ of real-valued functions on X is said to *separate points* in X provided for any two distinct points u and v in X, there is an f in $\mathcal{A}$ for which $f(u) \neq f(v)$.

Lemma. For X compact and Hausdorff, the whole algebra $C(X)$ of all real-valued functions separates points in X.

Proof. By Theorem 11.18, X is normal. By definition, a normal space is Tychonoff and by Proposition 11.6, singletons are closed sets in a Tychonoff space. Then by Urysohn's Lemma, for any two points $u, v \in X$ (since $\{u\}$ and $\{v\}$ are closed) and any interval $[a, b]$ $(a \neq b)$ there is a continuous $f : X \to \mathbb{R}$ such that $f(u) = a \neq b = f(v)$. Since $f \in C(X)$, then $C(X)$ separates points in X. ∎

Note. The topic of this section is a generalization of the Weierstrass Approximation Theorem. It is the following.

The Stone-Weierstrass Approximation Theorem.
Let X be a compact Hausdorff space. Suppose $\mathcal{A}$ is an algebra of continuous real-valued functions on X that separates points in X and contains the constant functions. Then $\mathcal{A}$ is dense in $C(X)$.

Note. The above result is a generalization of the Weierstrass Approximation as follows. We take $\mathcal{A}$ to be the collection of all polynomials on $[a, b]$. Then $\mathcal{A}$ is an algebra since a product of polynomials is a polynomial. Also, linear functions separate points. So the Stone-Weierstrass Theorem implies that $\mathcal{A}$ is dense in $C(X)$. (Notice that we cannot take the linear terms alone since a product of $mx + b$-type functions is not again of type $mx + b$.)

Note. The Stone-Weierstrass Theorem generalizes the Weierstrass Theorem and was first proved by Marshall Stone in 1937, hence the name. Before the proof, we need two preliminary lemmas.

Lemma 12.7. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions. Then for each closed subset F of X and point $x_0 \in X \sim F$, there is a neighborhood $\mathcal{U}$ of x_0 that is disjoint from F and has the following property: For each $\varepsilon > 0$ there is a function $h \in \mathcal{A}$ for which

$$h < \varepsilon \text{ on } \mathcal{U},\ h > 1 - \varepsilon \text{ on } F, \text{ and } 0 \leq h \leq 1 \text{ on } X. \tag{10}$$

Lemma 12.8. Let X be a compact Hausdorff space and $\mathcal{A}$ an algebra of continuous functions on X that separates points and contains the constant functions. Then for each pair of disjoint closed subsets A and B of X and $\varepsilon > 0$, there is a function h belonging to $\mathcal{A}$ for which

$$h < \varepsilon \text{ on } A,\ h > 1 - \varepsilon \text{ on } B, \text{ and } 0 \leq h \leq 1 \text{ on } X.$$

Note. We now have the equipment to prove the Stone-Weierstrass Theorem.

Note. We now state and prove a result showing that separability and metrizability are intimately related.

Borsuk's Theorem.

Let X be a compact Hausdorff topological space. Then $C(X)$ is separable if and only if X is metrizable.

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