Chapter 13. Continuous Linear Operators Between Banach Spaces

Note. In Chapters 13 to 15 we define "abstract" Banach spaces and consider continuous linear mappings between such spaces. In Chapter 16 we address Hilbert spaces.

Section 13.1. Normed Linear Spaces

Note. In this section, we define an abstract linear space and a norm on the space. We've seen examples of normed linear spaces with the $L^p(E)$ spaces in Chapters 7 and 8 and the space $C(X)$ of continuous real valued functions on compact Hausdorff space X in Section 12.3 (in addition to the familiar examples $\mathbb{R}^n$ from linear algebra).

Definition. A *linear space* is an abelian group X with group operation $+$ along with a *scalar product* mapping $\mathbb{R} \times X \to X$ where we denote the image of $(\alpha, u) \in \mathbb{R} \times X$ as $\alpha x \in X$, such that for all $\alpha, \beta \in \mathbb{R}$ and $u, v \in X$ we have

$(\alpha + \beta)u = \alpha u + \beta v,$

$\alpha(u + v) = \alpha u + \beta v,$

$(\alpha\beta)u = \alpha(\beta u)$ and $1u = u.$

The elements of X are *vectors*.

Definition. A *norm* on a linear space X, $\|\cdot\|$, is a nonnegative real valued function such that for all $u, v \in X$ and $\alpha \in \mathbb{R}$ we have

$\|u\| = 0$ if and only if $u = 0$,

$\|u + v\| \leq \|u\| + \|v\|$ (the Triangle Inequality),

$\|\alpha u\| = |\alpha|\|u\|$.

Note. A normed linear space has a metric on it defined as $d(u, v) = \|u - v\|$ (see Section 9.1). In a normed linear space, we discuss metric properties (such as boundedness, Cauchy sequences, and completeness) in terms of this metric and we discuss topological properties (such as open/closed, compact, or convergence) in terms of the induced metric topology (unless otherwise stated, such as when we encounter the "weak topology" on X in Section 14.1).

Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X are *equivalent* if there are constants $c_1 \geq 0$ and $c_2 \geq 0$ for which $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1$ for all $x \in X$.

Note. Two norms on X are equivalent if and only if their corresponding metrics are equivalent (see Section 9.1 for the definition of equivalent metrics).

Definition. Given finite collection of vectors $x_1, x_2, \ldots, x_n$ in linear space X and real scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$, the vector $x = \sum_{k=1}^n \lambda_k x_k$ is a *linear combination* of the x_i's. A nonempty subset Y of X is a *linear subspace* if Y is (algebraically) closed under linear combinations.

Definition. For $S \subset X$, the *span* of S is the set of all linear combinations of vectors in S, denoted $\text{span}[S]$. If $Y = \text{span}[S]$ then S *spans* Y. The (topological) closure of $\text{span}[S]$ in X, denoted $\overline{\text{span}}[S]$, is the *closed linear span* of S.

Note. In Exercise 13.3(ii) it is shown that $\text{span}[S]$ is a linear subspace of X which is the "smallest" subspace of X that contains S (since it is the intersection of all subspaces of X containing X). In Exercise 13.3(iii) it is shown that $\overline{\text{span}}[S]$ is a linear subspace of X which is the smallest (topologically) closed linear subspace containing S (since it is an intersection of all such subspaces).

Definition. For nonempty $A, B \subset X$, define the *sum* of A and B, denoted $A+B$, as $A + B = \{x + y \mid x \in A, y \in B\}$. If $B = \{x_0\}$ then $A + \{x_0\} = A + x_0$ is a *translate* of set A by *translation vector* x_0. For $\lambda \in \mathbb{R}$, $\lambda A = \{\lambda x \mid x \in A\}$. If Y and Z are subspaces of X then so is $Y + Z$ (by Exercise 13.2). If $X \cap Y = \{0\}$ then $Y + Z$ is the *direct sum* of X and Y, denoted $Y \oplus Z$.

Definition. In normed linear space X, the set $\{x \in X \mid \|x\| < 1\}$ is the *open unit ball* and the set $\{x \in X \mid \|x\| \leq 1\}$ is the *closed unit ball.* A vector $x \in X$ is a *unit vector* if $\|x\| = 1$.

Definition. A normed linear space is a *Banach space* if it is complete with respect to the metric induced by the norm.

Note. Of course we need completeness to do "analysis things"!

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