Section 13.2. Linear Operators

Note. In this section, we extend the idea of a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ to the setting of general linear spaces. In Section 8.1 we had a similar idea when mapping $L^p(E)$ to $\mathbb{R}$ with linear functionals.

Definition. Let X and Y be linear spaces. A mapping $T : X \to Y$ is *linear* provided that for each $u, v \in X$ and for any $\alpha, \beta \in \mathbb{R}$ we have $T(\alpha u + \beta u) = \alpha T(u) + \beta T(v)$. Such a mapping is also called a *linear operator*.

Definition. Let X and Y be normed linear spaces. A linear operator $T : X \to Y$ is *bounded* provide there is $M \geq 0$ for which $\|T(u)\| \leq M\|u\|$ for $u \in X$. The infimum of all such M is the *operator norm*:

$$\|T\| = \inf_{M \geq 0}\{\|T(u)\| \leq M\|u\| \text{ for all } u \in X\}.$$

The collection of bounded linear operators from X to Y is denoted $\mathcal{L}(X, Y)$.

Note. In Exercise 13.1, it is shown for $T \in \mathcal{L}(X, Y)$ that $\|T\| = \sup\{\|T(u)\| \mid u \in X, \|u\| \leq 1\}$.

Theorem 13.1 A linear operator between normed linear spaces is continuous if and only if it is bounded.

Note. We now define a linear combination of linear operators, thus making a new linear space of operators.

Definition. Let X and Y be linear spaces. For $T : X \to Y$ and $S : X \to Y$ linear operators and for $\alpha, \beta \in \mathbb{R}$, define $\alpha T + \beta S : X \to Y$ pointwise by $(\alpha T + \beta S)(u) = \alpha T(u) + \beta S(u)$ for all $u \in X$.

Proposition 13.2. Let X and Y be normed linear spaces. Then the collection of bounded linear operators from X to Y, $\mathcal{L}(X, Y)$, is itself a normed linear space.

Note. We now deal with the completeness of $\mathcal{L}(X, Y)$.

Theorem 13.3. Let X and Y be normed linear spaces. If Y is a Banach space, then so is $\mathcal{L}(X, Y)$.

Note. If $T \in \mathcal{L}(X, Y)$ is one to one and onto then T^{-1} is linear because for $v_1, v_2 \in X$ with $T(u_1) = v_1$ and $T(u_2) = v_2$ we have for $\alpha, \beta \in \mathbb{R}$ that

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2) - \alpha v_1 + \beta v_2,$$

so

$$T^{-1}(\alpha v_1 + \beta v_2) = \alpha u_1 + \beta u_2 = \alpha T^{-1}(v_1) + \beta T^{-1}(v_2),$$

so that T^{-1} is linear.

Definition. Let X and Y be linear spaces with $T \in \mathcal{L}(X, Y)$. If T is one to one, onto, and has a continuous inverse which is bounded (i.e., $T^{-1} \in \mathcal{L}(X, Y)$) then T is an *isomorphism*. If an isomorphism $T : X \to Y$ exists then X and Y are *isomorphic*. An isomorphism which preserves the norm, that is $\|T(u)\| = \|u\|$ for all $u \in X$, is an *isometric isomorphism*.

Definition. For linear $T : X \to Y$, the subspace of X, $\{x \in X \mid T(x) = 0\}$, is the *kernel* of T, denoted $\text{Ker}(T)$. The *image* of T is $\text{Im}(T) = T(X) = \{T(x) \mid x \in X\}$.

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