Section 13.3. Compactness Lost: Infinite Dimensional Normed Linear Spaces

Note. In this section, we give a few results for finite dimensional linear spaces. In particular, we prove that any n dimensional linear space (over $\mathbb{R}$) is isomorphic to $\mathbb{R}^n$ (in Corollary 13.5). We show in Corollary 13.7 that the closed unit ball in a finite dimensional space is compact (ultimately as a consequence of the Heine-Borel Theorem). The "compactness lost" part of the title of this section refers to Riesz's Theorem which states that the closed unit ball is compact in a linear space if and only if the space is finite dimensional.

Note. We take as given the usual ideas of *linear combination*, *linear independence*, *basis*, *span*, and *dimension* for a linear space as given in linear algebra.

Note. We will get a lot of mileage in the finite dimensional setting of the following result. In particular, we use it to classify finite dimensional normed linear spaces (Corollary 13.5), to show that finite dimensional normed linear spaces are complete (and so are examples of Banach spaces and Hilbert spaces), and to show that the closed unit ball is compact in any finite dimensional space (in Corollary 3.6). This last result has a sibling in the infinite dimensional case (Reiz's Theorem) which allows us to categorize the dimension of a linear space as finite or infinite in terms of the compactness or non-compactness of the closed unit ball.

Theorem 13.4. Any two norms on a finite dimensional linear space are equivalent.

Note. The following corollary to Theorem 13.4 shows that any two (real) normed linear spaces of the same dimension are isomorphic. Recall that any two vector spaces over the same scalar field and of the same dimension are isomorphic (this is what I call the "Fundamental Theorem of Finite Dimensional Vector Spaces"; see Theorem 5.1.2 of *Real Analysis with an Introduction to Wavelets*, Don Hong, Jianzhong Wang, and Robert Gardner, Academic Press/Elsevier Press, 2005). This result is slightly different; "linear space" and "vector space" are the some concept, but a vector space need not have a norm. However, in the case of an n dimensional vector space, we know that the space is isomorphic to $\mathbb{R}^n$ and so has the usual Euclidean norm.

Corollary 13.5. Any two normed linear spaces of the same finite dimension are isomorphic.

Corollary 13.6. Any finite dimensional normed linear space is complete and therefore any finite dimensional subspace of a normed linear space is closed.

Corollary 13.7. The closed unit ball in a finite dimensional normed linear space is compact.

Note. Corollary 13.7 foreshadows Riesz's Theorem, which classifies finite versus infinite dimensional spaces in terms of the compactness of the closed unit ball. The proof given here is the same as given in Fundamentals of Functional Analysis (MATH 5740); see my online notes for this on Section 2.8. Finite-Dimensional Normed Linear Spaces (notice Theorem 2.34).

Riesz's Lemma. Let Y be a (topologically) closed proper linear subspace of a normed linear space X. Then for each $\varepsilon > 0$ there is a unit vector $x_0 \in X$ for which $\|x_0 - y\| > 1 - \varepsilon$ for all $y \in Y$.

Riesz's Theorem. The closed unit ball of a normed linear space X is compact if and only if X is finite dimensional.

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