Section 14.2. The Hahn-Banach Theorem

Note. In this section we state and prove the Hahn-Banach Theorem. It involves extending a certain type of linear functional from a subspace of a linear to the whole space. It will ultimately give information about the dual space of the linear space. We'll see implications of the theorem in this section and throughout the remainder of this chapter.

Definition. A functional $p : X \to [0, \infty)$ on a linear space x is said to be *positively homogeneous* provided $[(\lambda x) = \lambda p(x)$ for all $x \in X$, $\lambda > 0$. It is *subadditive* provided $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Note. In Fundamentals of Functional Analysis (MATH 5740) a positively homogeneous subadditive functional is called a "Minkowski function" (see the online notes for Section 5.2: `http://faculty.etsu.edu/gardnerr/Func/notes/5-2.pdf`).

Definition. Let X be a linear space and let X_0 be a linear subspace. If X_0 has the property that there is some $x_0 \in X$, $x_0 \neq 0$, for which $X = X_0 + \text{span}\{x_0\}$, then X_0 is a linear subspace of *codimension* 1 in X. A translate of a linear subspace of codimension 1 is called a *hyperplane*.

Note. Any norm on linear space X is an example of a positively homogeneous subadditive functional.

Note. We need a preliminary lemma before proving out big result, the Hahn-Banach Theorem. The lemma involves an extension of a linear functional from a subspace to a space "one dimension large."

The Hahn-Banach Lemma. Let p be a positively homogeneous, subadditive functional on the linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y. Let z belong to $X \setminus Y$. Then ψ can be extended to a linear functional ψ on span$[Y + z]$ for which $\psi \leq p$ on span$[Y + z]$.

Note. The Hahn-Banach Theorem now allows us to extend ψ to all of X. The proof is the same given by Reed and Simon's *Functional Analysis I* (Academic Press, Methods of Modern Mathematical Physics, 1980) and Promislow's *A First Course in Functional Analysis* (Wiley, 2008) (though these references incorporate the Hahn-Banach Lemma into the proof of the theorem). It requires Zorn's Lemma.

Hahn-Banach Theorem. Let p be a positively homogeneous, subadditive functional on a linear space X and Y a subspace of X on which there is defined a linear functional ψ for which $\psi \leq p$ on Y. Then ψ may be extended to a linear functional ψ on all of X for which $\psi \leq p$ on all of X.

Note. The Hahn-Banach Theorem is named for Hans Hahn (1879–1934) and Stefan Banach (1892–1945). Hahn published the result for normed linear spaces in 1927 ("Über lineare Gleichungssysteme in linearen Raümen, *Journal für die Reine* *und Angewandte Math.* **157** 1927), 241–229). Banach proved an analytic version in 1932 ("Sur les fonctionelles Linéaires (II)," *Studia Mathematica* **4** (1933), 223–239). An earlier version was proved for the space $C([a,b])$ by Eduard Helly (1884–1943) in 1912. We will explore other versions in versions of the Hahn-Banach Theorem in this chapter. These notes are based on the Saint Andrews MacTutor History of Mathematics biography of Eduard Helly, and John Saccoman's "Evolution of the Geometric Hahn-Banach Theorem," *Rivista di matematica della Università di Parma*, **17** (1991), 257–264.

Hans Hahn (1879–1934)

Stefan Banach (1892–1945)

Theorem 14.7. Let X_0 be a linear subspace of a normed linear space X. Then each bounded linear functional ψ on X_0 has an extension to a bounded linear functional on all of X that has the same norm as ψ. In particular, for each $x \in X$ with $x \neq 0$ there is $\psi \in X^*$ for which $\psi(x) = \|x\|$ and $\|\psi\| = 1$.

Note. Theorem 14.7 shows that for every $x \in X$ there is $\psi \in X^*$, so as long as X is a nontrivial normed linear space then X^* contains nonzero bounded linear functionals (potentially, many).

Example. To illustrate how Theorem 14.7 can be used to establish the existence of elements of X^*, consider $X = L^\infty([a,b])$ and $X_0 = C([a,b])$ ($C([a.b])$ may be considered a subspace of $L^\infty([a,b])$ by Exercise 14.27). For any fixed $x_0 \in [a,b]$, define $\psi(f) = f(x_0)$ for all $f \in C([a,b])$). Then

$$\psi(\alpha f + \beta g) = (\alpha f + \beta g)(x_0) = \alpha f(x_0) + \beta g(x_0) = \alpha\psi(f) + \beta\psi(g),$$

so ψ is linear. Also, $\psi(f) = f(x_0) \leq \|f\||x_0|$, of ψ is bounded by $|x_0|$. That is, $\psi \in X^*$. By Theorem 14.7, ψ can be extended to a bounded linear functional defined on all of $X = L^\infty([a,b])$.

Example. Let $X = \ell^\infty$. Define $p(\{x_n\}) = \limsup\{x_n\}$ for all $\{x_n\} \in \ell^\infty$. Then

$$p(\lambda\{x_n\}) = p(\{\lambda x_n\}) = \limsup\{\lambda x_n\} = \lambda \limsup\{x_n\}$$

for $\lambda > 0$. Also

$$p(\{x_n\} + \{y_n\}) = p(\{x_n + y_n\}) = \limsup\{x_n + y_n\}$$

$$\leq \limsup\{x_n\} + \limsup\{y_n\} = p(\{x_n\}) + p(\{y_n\}).$$

So p is positively homogeneous and subadditive. Let $c_0 \subset$
ell^∞ be the subspace of convergent sequences. Define L on c_0 as $L(\{x_n\}) = \lim(x_n)$ for all $\{x_n\} \in c_0$. Then L is linear on c_0 and $L \leq p$ on c_0. So by Theorem 14.7, L has an extension to ℓ^∞ and the extension is $\leq p$ on ℓ^∞. Any such extension (we do not necessarily have uniqueness) is called a *Banach limit*.

Note. Recall that is Section 13.4 we defined the closed linear complement in a normed linear space. For X a normed linear space and V a subspace of X, a (topologically) closed subspace W of X is the closed linear complement of X if $X = V \oplus W$. The next result shows that a finite dimensional subspace of a normed linear space has a closed linear complement (recall by Corollary 13.6 that any finite dimensional subspace of a normed linear space is [topologically] closed).

Corollary 14.8. Let X be a normed linear space. If X_0 is a finite dimensional subspace of X, then there is a closed linear subspace X_1 of X for which $X = X_0 \oplus X_1$. That is, X_0 has a closed linear complement in X.

Corollary 14.9. Let X be a normed linear space. Then the natural embedding $J : X \to X^{**}$ is an isometry.

Theorem 14.10. Let X_0 be a subspace of the normed linear space X. Then a point $x \in X$ belongs to the closure of X_0 if and only if whenever a functional $\psi \in X^*$ vanishes on X_0, it also vanishes at x.

Corollary 14.11. Let $\mathcal{S}$ be a subset of the normed linear space X. Then the linear span of $\mathcal{S}$ is dense in X if and only if whenever $\psi \in X^*$ vanishes on $\mathcal{S}$, then $\psi = 0$.

Theorem 14.12. Let X be a normed linear space. Then every weakly convergent sequence in X is bounded. moreover, if $\{x_n\} \rightharpoonup x$ in X, then $\|x\| \leq \liminf \|x_n\|$.

Note. Royden and Fitzpatrick comment (page 281): "…often by making a clever choice of the functional p, [the Hahn-Banach Theorem] allows us to create basic analytical, geometric, and topological tools for functional analysis." We have seen that Theorem 14.7 (whose proof depends on the Hahn-Banach Theorem) allows us to find nonzero elements of X^*. In Section 14.4 we use the Hahn-Banach Theorem with p as the "gauge functional" associated with a convex set to separate disjoint convex subsets of a linear space by a hyperplane (this result is sometimes called the "geometric Hahn-Banach Theorem"). In Chapter 15 we use the natural embedding $J : X \to X^{**}$ to prove that the closed unit ball of a Banach space X is weakly sequentially compact if and only if X is reflexive (i.e., $J(X) = X^{**}$).

Revised: 5/1/2017