Chapter 16. Continuous Linear Operators on Hilbert Spaces

Note. In this chapter we consider Banach spaces for which the norm is induced by an inner product. Several of the results of the first three sections are also in *Real Analysis with an Introduction to Wavelets and Applications*, D. Hong, J. Wang, and R. Gardner, Elsevier Press (2005). the approach of Hong, Wang, and Gardner is to start with $\mathbb{R}^n$ and push the geometry of $\mathbb{R}^n$ to infinite dimensions.

Section 16.1. The Inner Product and Orthogonality

Note. We introduce inner product spaces where the "inner product" is similar to the dot product you see in sophomore linear algebra in $\mathbb{R}^n$. With the inner product, we can discuss orthogonality and projections.

Definition. Let H be a linear space. A function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ is an *inner product* on H if for all $x_1, x_2, x, y \in H$ and $\alpha, \beta \in \mathbb{R}$ we have

(i) $\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$,

(ii) $\langle x, y \rangle = \langle y, x \rangle$ (symmetry), and

(iii) $\langle x, x \rangle > 0$ if $x \neq 0$.

A linear space H together with an inner product is an *inner product space*. We call the elements of H *vectors*.

Note. Property (i) is called *linearity in the first component*. Combining (i) and (ii) gives *linearity in the second component*.

Examples. Linear space ℓ^2 is an inner product space where for $x = \{x_k\}_{k=1}^{\infty} \in \ell^2$ and $y = \|y_k\|_{k=1}^{\infty} \in \ell^2$ we define $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$. For measurable set E, $L^2(E)$ is an inner product space where for $f, g \in L^2(E)$ we define $\langle f, g \rangle = \int_E fg$.

Theorem. The Cauchy-Schwarz Inequality.

For any two vectors u and v in an inner product space H, $|\langle u, v \rangle| \leq \|u\| \|v\|$ where $\|u\| = \sqrt{\langle u, u \rangle}$.

Note. We have already seen the Cauchy-Schwarz Inequality for $L^2(E)$ in the setting of Hölder's Inequality.

Proposition 16.1. For a vector h in an inner product space H, define $\|h\| = \sqrt{\langle h, h \rangle}$. Then $\| \cdot \|$ is a norm on H called the *norm induced* by the inner product $\langle \cdot, \cdot \rangle$.

Note. The proof of Proposition 16.1 is almost identical to the proof seen in Linear Algebra (MATH 2010), but with the Cauchy-Schwarz Inequality stated as $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$.

Note. Recall (maybe from high school geometry) that in a parallelogram, the sum of the squares of the diagonals equals twice the sum of the squares of two adjacent sides. This is a property in $\mathbb{R}^2$ (and in Euclidean geometry in general). The next result shows that this property of parallelograms also holds in inner product spaces. This suggests that the geometry of an inner product space is Euclidean (and hence that such spaces are "flat"). This suspicion is confirmed in Section 16.3 where the Pythagorean Theorem (which is equivalent to the Parallel Postulate) is proved.

The Parallelogram Identity.

For any two vectors u, v in an inner product space H we have $\|u-v\|^2 + \|u+v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

Note. This being analysis we are, of course, interested in completeness.

Definition. An inner product space H is a *Hilbert space* if it is a Banach space with respect to the norm induced by the inner product.

Note. The Riesz-Fischer Theorem of Section 7.3, combined with the example above claiming that $L^2(E)$ is an inner product space, confirms that $L^2(E)$ is a Hilbert space. Similarly, Exercise 7.34 and the above example establishes that ℓ^2 is also a Hilbert space.

Proposition 16.2. Let K be a nonempty closed convex subset of a Hilbert space H and let $j \in H \setminus K$. Then there is exactly one vector $h_* \in K$ that is closest to h_0 in the sense that $\|h_0 - h_*\| = \text{dist}(h_0, K) = \inf_{h \in K} \|h_0 - h\|$.

Note. We now give a definition that will yield the most important property of an inner product.

Definition. Two vectors u, v in the inner product space H are *orthogonal* if $\langle u, v \rangle = 0$. A vector $u \in H$ is *orthogonal to subset* S of H if u is orthogonal to all $s \in S$. The set of vectors in H which are orthogonal to S is denoted $S^\perp$ (read "S perp").

Note. It is shown in Exercise 16.4 (using the Cauchy-Schwarz Inequality) that for $s \subset H$, $S^\perp$ is a closed subspace of H.

Note. Recall from Section 13.1 that if Y and Z are subspaces of a normed linear space X and $Y \cap Z = \{0\}$ then the *direct sum* of Y and Z is

$$Y \oplus Z = \{y + z \mid y \in Y \text{ and } z \in Z\}.$$

Proposition 16.3. Let V be a closed subspace of a Hilbert space H. Then H has the orthogonal direct sum decomposition $H = V \oplus V^\perp$.

Corollary 16.4. Let S be a subset of a Hilbert space H. Then the closed linear space of S is all of H if and only if $S^\perp = \{0\}$.

Definition. In Theorem 16.3, the set $v^\perp$ is the *orthogonal complement* of closed subspace V in H. The direct sum $H = V \oplus V^\perp$ is an *orthogonal decomposition* of H. The operator $P \in \mathcal{L}(H)$ that is the projection of H onto V is the *orthogonal projection* of H onto V.

Proposition 16.5. Let P be the orthogonal projection of a Hilbert space H onto a nontrivial closed subspace V of H. Then $\|P\| = 1$ and $\langle P(u), v\rangle = \langle u, P(v)\rangle$ for all $u, v \in H$.

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