Section 16.2. The Dual Space and Weak Sequential Convergence

Note. We now prove several of the results from Chapter 8, but in the general setting of Hilbert spaces (as opposed to the L^p space setting of Chapter 8).

Theorem. The Riesz-Fréchet Representation Theorem.
Let H be a Hilbert space. Define the operator $T : H \to H^*$ (where H^* is the dual space of H, the linear space of all bounded linear functionals on H) by assigning to each $h \in H$ the linear functional $T(h) : H \to \mathbb{R}$ defined by

$$T(h)[u] = \langle h, u \rangle \text{ for all } h \in H.$$

Then T is a linear isometry of H onto H^*.

Note. Recall that in a normed linear space X, sequence $\{f_n\}_{n=1}^{\infty} \subset X$ is weakly convergent to $f \in X$, denoted $\{f_n\} \rightharpoonup f$, if $\lim_{n\to\infty} T(f_n) = T(f)$ for all $T \in X^*$.

Definition. Let H be a Hilbert space and $\{u_n\}_{n=1}^{\infty} \subset H$. Then $\{u_n\}$ *converges weakly* to $u \in H$ if $\lim_{n\to\infty} T(u_n) = T(u)$ for all $T \in H^*$. We denote this as $\{u_n\} \rightharpoonup u$.

Note. By the Riesz-Fréchet Representation Theorem, $\{u_n\} \rightharpoonup u$ if and only if $\lim_{n\to\infty}\langle h, u_n\rangle \to \langle h, u\rangle$ for all $h \in H$.

Note. Recall that the Bolzano-Weierstrass Theorem states that every bounded sequence of real numbers (or, more generally, elements of $\mathbb{R}^n$) has a convergent subsequence. The following result is similar but in the setting of a Hilbert space.

Theorem 16.6. Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.

Note. The following is somewhat of a converse of Theorem 16.6. The proof is to be given in Exercise 16.17.

Proposition 16.7. Let $\{u_n\} \to u$ weakly (that is, $\{u_n\} \rightharpoonup u$) in Hilbert space H. Then $\{u_n\}$ is bounded and $\|u\| \leq \liminf \|u_n\|$. Moreover, if $\{v_n\| \to v$ (strongly) in H then $\lim_{n\to\infty}\langle u_n, v_n\rangle = \langle u, v\rangle$.

Note. A statement of the Banach-Saks Theorem was given in Section 8.4 in the setting of $L^p(E)$ for $1 < p < \infty$ (but in that section, the proof was only given in the case $p = 2$). We now give a statement and proof in the Hilbert space setting.

Theorem. The Banach-Saks Theorem.

Let $\{u_n\} \rightharpoonup u$ weakly in Hilbert space H. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ for which

$$\lim_{k\to\infty} \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} = u \text{ (strongly) in } H.$$

Note. Recall from Section 8.2, the Radon-Riesz Theorem in the $L^p(E)$ setting states: "Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then

$$\{f_n\} \to f \text{ in } L^p(E) \text{ if and only if } \lim_{n\to\infty} \|f_h\|_p = \|f\|_p.\text{"}$$

Royden and Fitzpatrick claim that neither this nor the Banach-Saks Theorem hold in general Banach spaces. We have seen that the Banach-Saks Theorem holds in Hilbert spaces and now we show that the Radon-Riesz Theorem holds in Hilbert spaces.

The Radon-Riesz Theorem.

Let $\{u_n\} \to u$ weakly (that is, $\{u_n\} \rightharpoonup u$) in the Hilbert space H. Then

$$\{u_n\} \to u \text{ strongly in } H \text{ if and only if } \lim_{n\to\infty} \|u_n\| = \|u\|.$$

Here, "strong convergence" means convergence with respect to the Hilbert space norm.

Note. The following result has a proof based on some heavy equipment! The proof, which we omit, uses Mazur's Theorem from Section 14.5, the Krein-Milman Theorem from Section 14.6, and Kakutani's Thereom from Section 15.2.

Theorem 16.8. Let H be a Hilbert space. Then H is reflexive. Therefore every nonempty strongly closed bounded convex subset k of H is weakly compact and hence is the strongly closed convex hull of its extreme points.