Section 16.3. Bessel's Inequality and Orthonormal Bases

Note. This section introduces many of the cool geometric properties of a Hilbert space! Much of the geometry and most of the vector space properties of $\mathbb{R}^n$ translates over to a certain class of Hilbert spaces (namely, those with an orthonormal basis).

Definition. A subset S of H is *orthogonal* provided every two vectors in S are orthogonal. If such a set has the further property that each vector in S is a unit vector, then S is *orthonormal*.

Note. The next two results are very similar to results from $\mathbb{R}^n$.

Theorem. The General Pythagorean Identity.

If $u_1, u_2, \ldots, u_n$ are n orthonormal vectors in H, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ then

$$\|\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n\|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2.$$

Theorem. Bessel's Inequality.

For $\{\varphi_k\}$ an orthonormal sequence in H and $h \in H$,

$$\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle^2 \leq \|h\|^2.$$

Note. The following is a step towards discussing a basis of a Hilbert space.

Proposition 16.9. Let $\{\varphi_k\}$ be an orthonormal sequence in a Hilbert space H and let $h \in H$. Then the series $\sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k$ converges strongly in H and the vector $h - \sum_{k=1}^{\infty} \langle \varphi_k, h \rangle \varphi_k$ is orthogonal to each φ_k.

Definition. An orthonormal sequence $\{\varphi_k\}$ in a Hilbert space H is *complete* provided the only vector $h \in H$ that is orthogonal to every φ_k is $h = 0$.

Lemma 16.3.A. Let $\{\varphi_k\}$ be an orthonormal sequence in Hilbert space H. Then $\{\varphi_k\}$ is complete if and only if the closed linear span of $\{\varphi_k\}$ is H.

Note. We finally define a "basis" for a Hilbert space.

Proposition 16.10. An orthonormal sequence $\{\varphi_k\}$ is a Hilbert space H is complete if and only if it is an orthonormal basis.

Note. If a Hilbert space has an orthonormal basis $\{\varphi_k\}$ then all finite linear combinations of elements of $\{\varphi_k\}$ with rational coefficients forms a countable dense set in H. That is, H is separable. The converse of this also holds (as shown in the next result) so that a Hilbert space is separable if and only if it has an orthonormal basis. Royden and Fitzpatrick defined a Hilbert space as an inner product space which is a Banach space under the inner product. They could have been more direct and could have defined a Hilbert space as a complete inner product space. This is the definition used, for example, in *Real Analysis with an Introduction to Wavelets*, Hong, Wang, and Gardner, Academic Press/Elsevier Press, 2005.

Theorem 16.11. Every infinite dimensional separable Hilbert space has an orthonormal basis.

Note. Royden and Fitzpatrick miss an opportunity here. It is straightforward at this point to define a Hilbert space isomorphism and prove that any two infinite dimensional Hilbert space with an orthonormal basis are isomorphic; in fact, they are isomorphic to ℓ^2. I call this the "Fundamental Theorem of Infinite Dimensional Vector Spaces" (see Theorem 5.4.9 of *Real Analysis with an Introduction to Wavelets*).

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