Chapter 2. Lebesgue Measure

Section 2.1. Introduction

**Note.** We “weigh” an interval by its length when setting up the Riemann integral. So to generalize the Riemann integral, we desire a way to weigh sets other than intervals. This weight should be a generalization of the length of an interval.

**Note.** Since we know an open set is a countable union of disjoint open intervals, we would define its “weight” (or “measure”) to be the sum of the lengths of the open intervals which compose it.

**Note.** We want a function $m$ which maps the collection of all subsets of $\mathbb{R}$, that is the power set of the reals $\mathcal{P}(\mathbb{R})$, into $\mathbb{R}^+ \cup \{0, \infty\} = [0, \infty]$. We would like $m$ to satisfy:

1. For any interval $I$, $m(I) = \ell(I)$ (where $\ell(I)$ is the length of $I$).
2. For all $E$ on which $m$ is defined and for all $y \in \mathbb{R}$, $m(E + y) = m(E)$. That is, $m$ is translation invariant.
3. If $\{E_k\}_{k=1}^\infty$ is a sequence of disjoint sets (on each of which, $m$ is defined), then $m(\bigcup E_k) = \sum m(E_k)$. That is, $m$ is countably additive.
4. $m$ is defined on $\mathcal{P}(\mathbb{R})$.

Here, and throughout, we use the symbol $\cup$ to indicate disjoint union.
Note. We will see in Section 2.6 that there is not a function satisfying all four properties. In fact, there is not even a set function satisfying (1), (2), and (4) for which \( m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k) \) for disjoint \( E_k \) (this property is called finite additivity). See Theorem 2.18 for details.

Note. It is “unknown” whether \( m \) exists satisfying properties (1), (3), and (4) (if we assume the Continuum Hypothesis, then there is not such a function).

Note. We will weaken Property (4) and try to find a function defined on as large a set as possible. We will require (by (3)) that our collection of sets, \( \mathcal{M} \), on which \( m \) is defined, be countably additive and therefore \( \mathcal{M} \) will be a \( \sigma \)-algebra.

**Problem 2.1.** Let \( m' \) be a set function defined on a \( \sigma \)-algebra \( \mathcal{A} \) with values in \([0, \infty] \). Assume \( m' \) is countably additive over countable disjoint collections in \( \mathcal{A} \). If \( A \) and \( B \) are two sets in \( \mathcal{A} \) with \( A \subset B \), then \( m'(A) \leq m'(B) \). This is called monotonicity.

Note. Another property of measure is the following.

**Problem 2.3.** Let \( \{E_k\}_{k=1}^{\infty} \) be a countable collection of sets in a \( \sigma \)-algebra \( \mathcal{A} \) on which a countably additive measure \( m' \) is defined. Then \( m'(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m'(E_k) \). This is called countable subadditivity.