Chapter 4. Lebesgue Integration

Section 4.1. Riemann Integral

**Note.** In this brief section, we define the Riemann integral of \( f \) on \([a, b]\) (when it exists) in two equivalent ways. The first way is probably what you saw in senior-level Analysis 2 (MATH 4227/5227) and the second way will act as inspiration of our approach to Lebesgue integration.

**Definition.** A *partition* of the interval \([a, b]\) is a set \( P = \{x_0, x_1, \ldots, x_n\} \subset [a, b] \) such that \( a = x_0 < x_1 < \cdots < x_n = b \).

**Definition.** Let \( f \) be a bounded function on \([a, b]\). For \( P \) a partition of \([a, b]\), the *lower and upper Darboux sums* for \( f \) with respect to \( P \) are

\[
L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \quad \text{and} \quad U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),
\]

respectively, where for \( 1 \leq i \leq n \)

\[
m_i = \inf\{ f(x) \mid x_{i-1} < x < x_i \} \quad \text{and} \quad M_i = \sup\{ f(x) \mid x_{i-1} < x < x_i \}.
\]
**Definition.** Let $f$ be a bounded function on $[a, b]$. The *lower and upper Riemann integrals* of $f$ over $[a, b]$ are

\[
(R) \int_a^b f = \sup \{ L(f, P) \mid P \text{ is a partition of } [a, b] \},
\]

and

\[
(R) \int_a^b f = \inf \{ U(f, P) \mid P \text{ is a partition of } [a, b] \}.
\]

**Definition.** Let $f$ be a bounded function on $[a, b]$. The $f$ is *Riemann integrable* on $[a, b]$ if \((R) \int_a^b f = (R) \int_a^b f\).

**Note.** These are the same definitions as used by J.R. Kirkwood in *An Introduction to Analysis*, 2nd Edition, Waveland Press (2002), with the exception that $m_i$ and $M_i$ are defined using $[x_{i-1}, x_i]$ and the term “Riemann sum” replaces the term “Darboux sum.” See my online notes for Analysis 2 (MATH 4227/5227) on 6.1. The Riemann Integral based on Kirkwood’s book, and the notes for this class on The Riemann-Lebesgue Theorem.

**Definition.** A real-valued function $\psi$ defined on $[a, b]$ is a *step function* if there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and numbers $c_1, c_2, \ldots, c_n$ such that for $1 \leq i \leq n$, $\psi(x) = c_i$ if $x_{i-1} < x < x_i$. 
4.1. The Riemann Integral

**Note.** Notice that \( L(\psi, P) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = U(\psi, P) \) for all partitions \( P \) of \([a, b]\). So \( \psi \) is Riemann integrable on \([a, b]\) and \( (R) \int_{a}^{b} \psi = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) \). So we can also express lower and upper Riemann integrals as

\[
(R) \int_{a}^{b} f = \sup \left\{ (R) \int_{a}^{b} \varphi \mid \varphi \text{ is a step function and } \varphi \leq f \text{ on } [a, b] \right\}
\]

and

\[
(R) \int_{a}^{b} f = \inf \left\{ (R) \int_{a}^{b} \varphi \mid \varphi \text{ is a step function and } f \leq \varphi \text{ on } [a, b] \right\},
\]

respectively. This idea of approximating \( f \) from below with “easy to integrate functions” and taking a supremum, and approximating \( f \) from above with “easy to integrate functions” and taking an infimum, will be the approach we take with Lebesgue integration in the next section.

*Revised: 10/31/2020*