Section 4.5. Countable Additivity and Continuity of Integration

Note. In this section, we prove two results for Lebesgue integrals which parallel results of Lebesgue measure. These properties “have no counterpart for the Riemann integral” (page 90), unlike many of the previous results of this chapter.

Theorem 4.20. The Countable Additivity of Integration.

Let $f$ be integrable over $E$ and $\{E_n\}_{n=1}^{\infty}$ a disjoint collection of measurable subsets of $E$ whose union is $E$. Then

$$\int_E f = \sum_{n=1}^{\infty} \left( \int_{E_n} f \right).$$

Theorem 4.21. The Continuity of Integration.

Let $f$ be integrable over $E$.

(i) If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of $E$ (that is, $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$), then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \int_{\lim_{n \to \infty} E_n} f = \lim_{n \to \infty} \left( \int_{E_n} f \right).$$

(ii) If $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of $E$ (that is, $E_{i+1} \subset E_i$ for all $i \in \mathbb{N}$), then

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \int_{\lim_{n \to \infty} E_n} f = \lim_{n \to \infty} \left( \int_{E_n} f \right).$$

Note. The proof of Theorem 4.21 is Exercise 4.39.

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