Chapter 9. Metric Spaces: General Properties

Note. In this chapter, we consider "abstract" metric spaces, which consist simply of a set of points X and a metric which gives an idea of the distance between any two elements of X. The existence of a metric lets us study all analysis ideas including open sets, closed sets, compact sets, limits, continuity, completeness, and separability. This is similar to our study of normed linear spaces, but we assume no linear structure in a metric space.

Section 9.1. Examples of Metric Spaces

Note. In this section we give the basic definitions of metric spaces and present a few examples.

Definition. Let X be a nonempty set. A function $\rho : X \times X \to \mathbb{R}$ is a *metric* provided for all $x, y, z \in X$ we have

(i) $\rho(x, y) \geq 0$.

(ii) $\rho(x, y) = 0$ if and only if $x = y$.

(iii) $\rho(x, y) = \rho(y, x)$.

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (The Triangle Inequality).

Set X together with metric ρ is a *metric space*, denoted (X, ρ).

Example. Of course $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$ is a metric space.

Example. If X is a normed linear space with norm $\|\cdot\|$ then X is a metric space where $\rho(x, y) = \|x - y\|$. Therefore normed linear space $\mathbb{R}^n$ under the usual Euclidean norm, $L^p(E)$ under the L^p norm, and $C([a, b])$ under the max norm are all metric spaces.

Example. For any nonempty set X, the discrete metric ρ defined by $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$ is a metric space. Therefore every nonempty set can have a metric on it. Similarly, any linear space X has a norm on it, say $\|x\| = \rho(0, x)$ where 0 is the zero vector in X and ρ is the discrete metric.

Example. For metric space (X, ρ), let Y be a nonempty subset of X. Then the restriction of ρ to $Y \times Y$ defines a metric on Y. (Y, ρ) is a *metric subspace* of (X, ρ).

Example. For metric spaces (X, ρ_1) and (X_2, ρ_2), define the *metric product* τ on $X_1 \times X_2$ as $\tau((x_1, x_2), (y_1, y_2)) = \left\{\rho_1(x_1, y_1)^2 + \rho_2(x_2, y_2)^2\right\}^{1/2}$. By Exercise 9.10, $(X_1 \times X_2, \tau)$ is a metric space (in fact, Exercise 9.10 shows that this idea can be extended to a countable product, with sufficient care).

Definition. Two metrics ρ and σ on a set X are *equivalent* if there are positive numbers c_1 and c_2 such that for all $x_1, x_2 \in X$, $c_1\sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2\sigma(x_1, x_2)$.

Note. Metric spaces have little structure so the idea of an "isomorphism" in the metric space setting is fairly weak, as seen in the following definition.

Definition. A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is an *isometry* if it maps X onto Y and for all $x_1, x_2 \in X$, $\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2)$. When such an isometry exists, (X, ρ) and (Y, σ) are *isometric*.

Note. An isometry is one to one since $x_1 \neq x_2$ implies $\rho(x_1, x_2) \neq 0$ and hence $\sigma(f(x_1), f(x_2)) \neq 0$ and $f(x_1) \neq f(x_2)$.

Definition. Let X be a nonempty set. A function $\rho : X \times X \to \mathbb{R}$ is a *pseudometric* if for all $x, y, z \in X$ we have

(i) $\rho(x, y) \geq 0$.

(ii) $\rho(x, y) = \rho(y, x)$.

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (The Triangle Inequality).

Set X together with pseudometric ρ is a *pseudometric space*, denoted (X, ρ).

Note. In a pseudometric space (X, ρ) we can define an equivalence relation $x \cong y$ provided $\rho(x, y) = 0$. We can then define a metric on the space of equivalence classes as $\tilde{\rho}([x], [y]) = \rho(x, y)$. We denote the resulting metric space of equivalence classes as $(X/\cong, \tilde{\rho})$. This is how we dealt with the L^p spaces where the equivalence classes consistent of functions equal a.e.