Section 9.2. Open Sets, Closed Sets, and Convergent Sequences

Note. In this section we introduce several topological ideas in a metric space.

Definition. Let (X, ρ) be a metric space. For a point $x \in X$ and $r > 0$, the set $B(x, r) = \{x' \in X \mid \rho(x', x) < r\}$ is the *open ball* centered at x of radius r. A subset $\mathcal{O}$ of X is *open* provided for every $x \in \mathcal{O}$ there is some open ball centered at x that is contained in $\mathcal{O}$. For a point $x \in X$, an open set that contains x is a *neighborhood* of x.

Note. An open ball is in fact an open set. Let $x' \in B(x, r)$. Define $r' = r - \rho(x', x)$. For $y \in B(x', r')$ we have $\rho(y, x) \leq \rho(y, x') + \rho(x', x) < r' + \rho(x', x) = r$ and so $y \in B(x, r)$. Therefore $B(x', r') \subset B(x, r)$ and so $B(x, r)$ is open by definition.

Proposition 9.1. Let X be a metric space. Then sets X and $\varnothing$ are open. The intersection of any two open subsets of X is open. The union of any collection of open subsets is open.

Note. Of course induction gives that the intersection of any finite number of open sets is open.

Proposition 9.2. Let X be a subspace of the metric space Y and E a subset of X. Then E is open in X if and only if $E = X \cap \mathcal{O}$ where $\mathcal{O}$ is open in Y.

Proof. The proof is to be given in Exercise 9.16.

Definition. For a subset E of a metric space X, a point $x \in X$ is a *point of closure* of E provided every neighborhood of x contains a point in E. The collection of points of closure of E is the *closure* of E and is denoted $\overline{E}$.

Note. You may be used to the definition of a closed set as the complement of an open set. This is not the definition here, but will ultimately be a result (Proposition 9.4).

Definition. For a subset E of a metric space X, E is *closed* if it contains all of its points of closure (that is, if $E = \overline{E}$). For $x \in X$ and $r > 0$, the set $\overline{B}(x, r) = \{x' \in X \mid \rho(x', x) \le r\}$ is the *closed ball* centered at x of radius r. If X is a metric space, then $B(0, 1)$ is the *open unit ball* and $\overline{B}(0, 1)$ is the *closed unit ball.*

Proposition 9.3. For E a subset of a metric space X, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed subset of X containing E in the sense that if F is closed and $E \subset F$ then $\overline{E} \subset F$.

Proposition 9.4. A subset of a metric space X is open if and only if its complement in X is closed.

Note. Of course Proposition 9.4 implies that a set is closed if and only if its complement is open. By De Morgan's Laws, Proposition 9.1 now implies the following.

Proposition 9.5. Let X be a metric space. Then sets X and $\varnothing$ are closed. The intersection of any collection of closed subsets is closed.

Definition. A sequence $\{x_n\}$ in a metric space (X, ρ) *converges* to $x \in X$ provided $\lim_{n\to\infty} \rho(x_n, x) = 0$. That is, for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n \geq N$, $\rho(x_n, x) < \varepsilon$. The point to which the sequence converges is the *limit* of the sequence and we write $\{x_n\} \to x$ to denote the convergence of $\{x_n\}$ to x.

Note. In a metric space, the limit of a sequence is unique (suppose $\{x_n\} \to x$ and $\{x_n\} \to y$; consider $\varepsilon = \rho(x, y)/2$). This may not be the case in a more general setting such as in a topological space.

Proposition 9.6. For a subset E of a metric space X, a point $x \in X$ is a point of closure of E if and only if x is the limit of a subsequence in E. Therefore, E is closed if and only if whenever a sequence in E converges to a limit $x \in X$, the limit of x belongs to E.

Proposition 9.7. Let ρ and σ be equivalent metrics on a nonempty set X. Then a subset of X is open in the metric space (X, ρ) if and only if it is open in the metric space (X, σ).

Proof. The proof is to be given in Exercise 9.17.

Note. Proposition 9.7 implies that equivalent metrics induce the same topology. This means that sets have the same closures, sequences have the same limits with respect to the two limits.

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