# Section 9.3. Continuous Mappings Between Metric Spaces

**Note.** We give an $\varepsilon/\delta$ property of continuity using metrics. An alternative approach to continuity in terms of open sets will be given and used in the topological space setting later as a definition.

**Definition.** A mapping $f$ from a metric space $X$ to a metric space $Y$ is *continuous* at point $x \in X$ provided for any sequence $\{x_n\}$ in $X$, we have that if $\{x_n\} \to x$ then $\{f(x_n)\} \to f(x)$. Mapping $f$ is *continuous* on $X$ provided it is continuous at every point $X$.

**Note 9.3.A.** We can treat the metric $\rho$ on metric space $X$ as a function $f(x, y)$ from $X \times X$ into $\mathbb{R}$ defined as $f(x, y) = \rho(x, y)$. To address continuity, we need a metric on $X \times X$; we take $\rho_{\max}((x_1, y_1), (x_2, y_2)) = \max\{\rho(x_1, x_2), \rho(y_1, y_2)\}$. Let $(x, y)$ be a point in $X \times X$ and let $\{(x_n, y_n)\}$ be any sequence in $X \times X$ such that $\{(x_n, y_n)\} \to (x, y)$ (with respect to $\rho_{\max}$). Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\rho_{\max}((x_n, y_n), (x, y)) < \varepsilon$. Since both $\rho(x_n, x) \leq \rho_{\max}((x_n, y_n), (x, y))$ and $\rho(y_n, y) \leq \rho_{\max}((x_n, y_n), (x, y))$, then for all $n \geq N$ we have both $\rho(x_n, x) < \varepsilon$ and $\rho(y_n, y) < \varepsilon$. Therefore, $\{x_n\} \to x$ and $\{y_n\} \to y$. By Exercise 9.14, this implies that $\rho(x_n, y_n) \to \rho(x, y)$. That is, if $\{(x_n, y_n)\} \to (x, y)$ in $X \times X$ then $f(x_n, y_n) = \rho(x_n, y_n) \to \rho(x, y)$. So, by definition, metric $\rho$ is a continuous function on $X \times X$.

**Note.** The following is likely more familiar to you as the definition of continuity. In fact, with the usual metric on $\mathbb{R}$ the $\varepsilon/\delta$ criterion for continuity, the property is equivalent to the Calculus 1 $\varepsilon/\delta$ definition of continuity.

**Theorem 9.3.A. The $\varepsilon/\delta$ Criterion for Continuity.**
A mapping $f$ from a metric space $(X, \rho)$ to a metric space $(Y, \sigma)$ is continuous at the point $x \in X$ if and only if for every point $\varepsilon > 0$ there is $\delta > 0$ for which if $\rho(x, x') < \delta$ then $\sigma(f(x), f(x')) < \varepsilon$; that is, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.

**Note.** When dealing with topological spaces in Chapter 11 (which may not have a metric), we will see that a continuous function satisfies the following property in that setting, as well as in our current metric space setting. See Proposition 11.10 in Section 11.4. Continuous Mappings Between Topological Spaces.

**Proposition 9.8.** A mapping $f$ from metric space $X$ to metric space $Y$ is continuous if and only if for each open subset $\mathcal{O}$ of $Y$, the inverse image under $f$ of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is an open subset of $X$.

**Note.** The next result is not surprising, given our experience with continuous real-valued functions of a real variable. We'll see this result also holds in the topological space setting. See Proposition 11.10 in Section 11.4.

**Proposition 9.9.** The composition of continuous mappings between metric spaces, when defined, is continuous.

**Note.** Next, we push two more ideas related to continuity from the real setting to the metric space setting.

**Definition.** A mapping from a metric space $(X, \rho)$ to a metric space $(Y, \sigma)$ is *uniformly continuous* on $X$ provided for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $u, v \in X$

$$\text{if } \rho(u, v) < \delta \text{ then } \sigma(f(u), f(v)) < \varepsilon.$$

**Definition.** A mapping $f$ from a metric space $(X, \rho)$ to a metric space $(Y, \sigma)$ is *Lipschitz* provided there is $c \geq 0$ such that for all $u, v \in X$, $\sigma(f(u), f(v)) \leq c\rho(u, v)$.

**Note.** A uniformly continuous function on $X$ is pointwise continuous on $X$ (but not conversely). A Lipschitz function is uniformly continuous (use the definition of uniformly continuous with $\delta = \varepsilon/c$). If a real valued function of a real variable is differentiable, then it is Lipschitz.

**Note.** For real valued functions of a real variable, we can set up a chain of subsets starting with continuous functions which are a subset of Lipschitz functions which are a subset of continuously differentiable functions which are a subset of "degree two Lipschitz" functions which are a subset of functions with a continuous second derivative which.... See my online notes for Complex Analysis 1 (MATH 5510) on Supplement. A Primer on Lipschitz Functions.

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