Section 9.4. Complete Metric Spaces

Note. Royden and Fitzpatrick declare "the structure of a metric space is too barren [by itself] to be fruitful in the study of interesting problems in mathematical analysis." See page 193. In this section we consider the property of completeness defined in terms of Cauchy sequences. We need completeness to do "analysis things" like limits and continuity.

Note. Recall that the Axiom of Completeness for the real numbers states that "Every set of real numbers with an upper bound has a least upper bound." The real numbers are then defined to be the complete ordered field. See my online notes for Analysis 1 (MATH 4217/5217) on Section 1.2. Properties of the Real Numbers as an Ordered Field and Section 1.3. The Completeness Axiom. Since we have no ordering in a metric space, we cannot use the least upper bound approach to completeness in a metric space (we cannot even use it in the complex setting since the complex numbers do not form an ordered field).

Note. Recall that a sequence of real numbers is convergent if and only if it is Cauchy; see Exercise 2.3.13 in my Analysis 1 notes on Section 2.3. Bolzano-Weierstrass Theorem. This result (the proof of which requires the Axiom of Completeness) inspires our definition of completeness. This is the same approach to completeness we took in normed linear spaces in Section 7.3. L^p is Complete: The Riesz-Fischer Theorem.

Definition. A sequence $\{x_n\}$ in a metric space (X, ρ) is a *Cauchy sequence* if for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ for which

$$\text{if } m, n \geq N \text{ then } \rho(x_n, x_m) < \varepsilon.$$

Definition. A metric space X is *complete* if every Cauchy sequence in X converges to a point in X.

Note. The real numbers form a metric space with the usual metric and under this metric, a sequence or real numbers is Cauchy if and only if it is convergent (see Exercise 2.3.13 from Analysis 1, as mentioned above). So the previous definition is consistent with our first exposure to completeness in $\mathbb{R}$.

Proposition 9.10. Let $[a, b]$ be a closed, bounded interval of real numbers. Then $C([a, b])$, with the metric induced by the max norm, is complete.

Note. We have not previously considered subspaces of complete spaces. A quick observation shows that a metric subspace of a complete space may not be complete. For example, an open bounded interval of the real numbers is not complete, but $\mathbb{R}$ is. Also, $\mathbb{Q}$ is a metric subspace of $\mathbb{R}$, but $\mathbb{Q}$ is not complete. The next result gives a classification of complete metric subspaces of a complete metric space.

Proposition 9.11. If E is a subset of the complete metric space X, then the metric subspace E is complete if and only if E is a closed subset of X.

Note. Since $\mathbb{R}^n$, $L^p(E)$ for $1 \leq p \leq \infty$, and $C[a, b]$ are all complete, then Proposition 9.11 implies the following.

Proposition 9.12. The following are complete metric spaces:

(i) Each nonempty closed subset of Euclidean space $\mathbb{R}^n$.

(ii) For E a measurable set of real numbers and $1 \leq p \leq \infty$, each nonempty closed subset of $L^p(E)$.

(iii) Each nonempty closed subset of $C[a, b]$.

Note. We now use a metric to define the diameter of a set. You will notice that this is very similar to the definition of the diameter of a graph in graph theory; see my online notes for Graph Theory 1 (MATH 5340) on Section 3.1. Walks and Connection. However, in a connected graph we use a maximum distance and here we use a supremum. So we may have sets of infinite diameter, since a supremum can be ∞.

Definition. For a nonempty subset E of a metric space (X, ρ), the *diameter* of E, denoted $\text{diam}(E)$, as

$$\text{diam}(E) = \sup\{\rho(x, y) \mid x, y \in E\}.$$

Set E is *bounded* provided it has a finite diameter. A descending sequence $\{E_m\}_{n=1}^{\infty}$ of nonempty subset of X is a *contracting sequence* provided $\lim_{n\to\infty} \text{diam}(E_n) = 0$.

Note. We now characterize complete metric spaces in terms of contracting sequences of nonempty closed sets.

Theorem 9.4.A. The Cantor Intersection Theorem. Let X be a metric space. Then X is complete if and only if whenever $\{F_n\}_{n=1}^\infty$ is a contracting sequence of nonempty closed subsets of X, there is a point $x \in X$ for which $\cap_{n=1}^\infty F_n = \{x\}$.

Note. We might think of the Cantor Intersection Theorem as implying that a metric space that is not complete has "holes" in it. If a contracting sequence of nonempty closed subsets of X is empty, then the hole is where the point x would be if the space were complete and formed a continuum. This is the situation, for example, for $\mathbb{Q}$ as a subspace of $\mathbb{R}$. The rational numbers $\mathbb{Q}$ has a hole at $\sqrt{2}$, and this is revealed by considering the contracting sequence of closed nonempty sets $F_n = (\sqrt{2} - 1/n, \sqrt{2} + 1/n) \cap \mathbb{Q}$ (these are in fact closed sets in metric space $\mathbb{Q}$). We have $\cap_{i=1}^n F_n = \varnothing$ in metric space $\mathbb{Q}$. Of course we could plug this hole by simply adding $\sqrt{2}$ and considering the metric space $\mathbb{Q} \cup \{\sqrt{2}\}$. But then there are other holes... If we plug all of the holes, then we get $\mathbb{R}$. The next result (the proof of which is outlined in Problem 9.49) shows that all of the holes of an incomplete metric space can be plugged.

Theorem 9.13. Let (X, ρ) be a metric space. Then there is a complete metric space $(\tilde{X}, \tilde{\rho})$ for which X is a dense subset of $\tilde{X}$ and

$$\rho(u, v) = \tilde{\rho}(u, v) \text{ for all } u, v \in X.$$

Definition. The metric space $(\tilde{X}, \tilde{\rho})$ is the *completion* of metric space (X, ρ).

Note. Any two completions of a metric space are isometric by way of an isometry that is the identity mapping on X. This is to be proved in Problem 9.50.

Note. A proof of Theorem 9.13 in the setting of normed linear spaces is given in Fundamentals of Functional Analysis (MATH 5740); see my online notes for this class on Section 2.5. Completeness and notice the Completion Theorem (Theorem 2.22).

Revised: 4/25/2023