Section 9.5. Compact Metric Spaces

Note. In this section, we define the concepts of compactness, boundedness, total boundedness, sequential compactness, and relate these ideas to each other. We extend several results concerning these concepts from the setting of $\mathbb{R}$ to metric spaces.

Definition. A collection of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ is a *cover* of set E if $E \subseteq \cup_{\lambda\in\Lambda} E_\lambda$. A *subcover* of a cover of E is a subcollection of the cover which itself is also a cover of E. If E is a subset of a metric space X, an *open cover* of E is a cover of E consisting of open subsets of X.

Definition. A metric space X is *compact* if every open cover of X has a finite subcover. A subset K of X is a *compact subset* if K, considered as a metric subspace of X, is compact.

Definition. An *open subset of subspace* K of metric space X is the intersection of K with an open subset of X.

Note 9.5.A. An open subset of the metric subspace K of metric space X is the intersection of K with an open subset of X. So subset K of metric space X is compact if and only if each cover of K by a collection of open subsets <u>of X</u> has a finite subcover. So the definition of a compact set in the metric space setting is ultimately the same as the definition in the setting of $\mathbb{R}$ (see, for example, my online notes for Analysis 1 [MATH 4217/5217] on Section 3.1. Topology of the Real Numbers).

Note 9.5.B. If $\mathcal{T}$ is a collection of subsets of a metric space which form a covering of X, that is $X \subset \cup_{E \in \mathcal{T}} E$, then by De Morgan's Law

$$\varnothing = X^c = (\cup_{E \in \mathcal{T}} E)^c = \cap_{E \in \mathcal{T}} E^c.$$

If $\mathcal{T}$ is a collection of open subsets in metric space X then the collection $\mathcal{F}$ of complements of sets in $\mathcal{T}$ is a collection of closed sets. Therefore, metric space X is compact if and only if every collection of closed sets with a nonempty intersection has a finite subcollection whose intersection is also nonempty.

Definition. A collection $\mathcal{F}$ of sets in X has the *finite intersection property* if any subcollection of $\mathcal{F}$ has a nonempty intersection.

Note. From the observation of Note 9.5.B and the definition of the finite intersection property, we have the following classification of compact sets.

Proposition 9.14. A metric space X is compact if and only if every collection $\mathcal{F}$ of closed subsets of X with the finite intersection property has nonempty intersection.

Definition. A metric space X is *totally bounded* provided for each $\varepsilon > 0$, the space X can be covered by a finite number of open balls of radius ε. A subset E of X is *totally bounded* provided that E, considered as a metric subspace of the metric space X, is totally bounded.

Definition. Let E be a subset of metric space X. An *ε-net* for E is a finite collection of open balls $\{B(x_k, \varepsilon)\}_{k=1}^n$ with centers $x_k \in X$ whose union covers E.

Note 9.5.C. In Problem 9.57 it is to be shown that the metric space E is totally bounded if and only if for each $\varepsilon > 0$, there is a finite ε-net for E. Notice we then have that a subset E of metric space X can be shown to be totally bounded by showing that there is an ε-net of E in X. In this way, the centers of the balls $\{B(x_k, \varepsilon)\}_{k=1}^n$ need not be in E, but instead can have centers in X (this is the point of Problem 9.57).

Lemma 9.5.A. If a metric space X is totally bounded then it is bounded in the sense that its diameter is finite.

Note. The converse of Lemma 9.5.A does not holds, as the next example shows.

Example 9.5.A. Let X be the Banach space ℓ^2 of square summable sequences. Consider the closed unit ball $B\{\{x_n\} \in \ell^2 \mid \|\{x_n\}\|_2 \leq 1\}$. Then B is bounded with diameter 2 (by the Triangle Inequality). However, as we show next, B is not totally bounded. Let $n \in \mathbb{N}$, let $e_n \in \ell^2$ have nth component 1 and other components 0. Then $\|e_n - e_m\|_2 = \sqrt{2}$ for $m \neq n$. So B cannot be contained in a finite number of balls of radius $r < 1/2 < 1/\sqrt{2}/2$ since one of these balls would contain two of the e_n's, which are distance $\sqrt{2}$ apart and yet the ball has diameter less than 1.

Note. The L^p and ℓ^p spaces are examples of linear spaces for $1 \leq p \leq \infty$. In Section 13.3. Infinite Dimensional Normed Linear Spaces we'll see that a normed linear space is finite dimensional if and only if the closed unit ball in the space is compact (see Riesz's Theorem). This is also shown in Fundamentals of Functional Analysis (MATH 5740); see my online notes for Fundamentals of Functional Analysis on Section 2.8. Finite Dimensional Normed Linear Spaces; notice Theorem 2.34 (Riesz's Theorem).

Proposition 9.15. A subset of Euclidean space $\mathbb{R}^n$ is bounded if and only if it is totally bounded.

Note. Just as we now have two different kinds of boundedness, we introduce a second type of compactness. We tie these ideas together below.

Definition. A metric space X is *sequentially compact* if every sequence in X has a subsequence that converges to a point in X.

Theorem 9.16. Characterization of Compactness for a Metric Space. For a metric space X, the following three assertions are equivalent:

(i) X is complete and totally bounded;

(ii) X is compact;

(iii) X is sequentially compact.

Note. We break the proof of Theorem 9.16 into three pieces.

Proposition 9.17. If a metric space X is complete and totally bounded, then it is compact.

Proposition 9.18. If a metric space X is compact, then it is sequentially compact.

Proposition 9.19. If a metric space X is sequentially compact, then it is complete and totally bounded.

Note. We know that $\mathbb{R}^n$ is complete, and by Proposition 9.11 each closed subset of $\mathbb{R}^n$ is complete as a metric subspace. By Proposition 9.15 a subset of $\mathbb{R}^n$ is bounded if and only if it is totally bounded. These observations, along with Characterization of Compactness for a Metric Space (Theorem 9.16), imply the following about a subset K of $\mathbb{R}^n$.

Theorem 9.20. For a subset K of $\mathbb{R}^n$, the following three assertions are equivalent:

(i) K is closed and bounded;

(ii) K is compact;

(iii) K is sequentially compact.

Note. The equivalence of 'closed and bounded' and compactness in $\mathbb{R}^n$ (i.e., the equivalence of (i) and (ii) in Theorem 9.20) is the familiar Heine-Borel Theorem. The equivalence of 'closed and bounded' and sequential compactness in $\mathbb{R}^n$ (i.e., the equivalence of (i) and (iii) in Theorem 9.20) is the familiar Bolzano-Weierstrass Theorem. For details on these results in $\mathbb{R}^1$, see my online notes for Analysis 1 (MATH 4217/5217) on Section 3.1. Topology of the Real Numbers (the Heine-Borel Theorem is given as Theorem 3-10/3-11) and Section 2.3. Bolzano-Weierstrass Theorem (notice Theorem 2-12). The next result is also encountered in Analysis 1 in the setting of the real numbers (see Theorem 4-7 of Section 4.1. Limits and Continuity).

Proposition 9.21. Let f be a continuous mapping from a compact metric space X to a metric space Y. Then its image $f(X)$ is also compact.

Note. In fact, Proposition 9.21 also holds in the topological space setting (see Proposition 11.20 of Section 11.5. Compact Topological Spaces; in fact, the next proposition is given as a corollary to Proposition 9.21 in this Section 11.5 also). We might paraphrase Proposition 9.21 as "a continuous function preserves compactness." Another property preserved by a continuous function is connectivity. We do not address connectivity in the metric space setting, but we will see it in the topological setting. In Proposition 11.22 of Section 11.6. Connected Topological Spaces it is shown that a continuous function maps a connected set to a connected set. That is, the property of connectivity is preserved by a continuous function.

In fact, *this* is the reason a continuous function is called "continuous" (it is easy to loose sight of the intuitive meaning of continuity after being exposed to years of limits, ε's, and δ's). You can remember these properties because of the use of the letter "**c**." A **c**ontinuous function preserves **c**ompactness and **c**onnectedness. Beware, that the property of being a closed set is not preserved by a continuous function!

Note. The next result is familiar to you from Calculus 1 (MATH 1910). It is used to justify the search for extrema of a continuous function on an interval of the form $[a, b]$; see my online Calculus 1 notes on Section 4.1. Extreme Values of Functions on Closed Intervals (notice The Extreme-Value Theorem for Continuous Functions, Theorem 4.1).

Proposition 9.22. Extreme Value Theorem.
Let X be a metric space. Then X is compact if and only if every continuous real-valued function on X takes a maximum and a minimum value.

Definition. If $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is an open cover of a metric space X, then each $x \in X$ is contained in some $\mathcal{O}_\lambda$ and, since $\mathcal{O}_\lambda$ is open, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \mathcal{O}_\lambda$. If there is a $\varepsilon > 0$ such that this holds independent of the choice of $x \in X$, then ε is the *Lebesgue number* for the cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ of X.

Note. The next result shows that a compact metric space has, for each given open cover $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ of X, a Lebesgue number.

The Lebesgue Covering Lemma. Let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be an open cover of a compact metric space X. Then there is a number $\varepsilon > 0$, such that for each $x \in X$, the open ball $B(x, \varepsilon)$ is contained in some member of the cover.

Note. Our final result of this section also holds in $\mathbb{R}$. In Analysis 1 (MATH 4217/5217), Corollary 4-10 states that a continuous real-valued function on a compact set of real numbers is uniformly continuous (see Section 4.1. Limits and Continuity).

Proposition 9.23. A continuous mapping from a compact metric space (X, ρ) into a metric space (Y, σ) is uniformly continuous.

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