Section 9.6. Separable Metric Spaces

Note. We first encountered the ideas of a dense subset and a separable space in Section 7.4. Approximation and Separability, where we defined these for normed linear spaces. We know that the normed linear spaces $L^p(E)$ are examples of metric spaces (where the distance between two elements of $L^p(E)$ is the norm of the difference of the elements). In this section, we extend the definition of a dense subset and a separable space to general metric spaces.

Definition. A subset D of a metric space X is *dense* in X provided every nonempty open subset of X contains a point of D. A metric space X is *separable* provided there is a countable subset of X that is dense in X.

Note 9.6.A. By the definition of "point of closure," set D is dense in X if and only if every point in X is a point of closure of D; that is, $\overline{D} = X$. Notice that $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable dense subset (similarly, $\mathbb{R}^n$ is separable since $\mathbb{Q}^n$ is a countable dense subset). The Weierstrass Approximation Theorem of Section 12.3. The Stone-Weierstrass Theorem implies that the polynomials are dense in $C[a, b]$; the polynomials with rational coefficients are in fact dense in $C[a, b]$, so that $C[a, b]$ is separable. By Theorem 7.11, shows that $L^p(E)$ is separable for for $1 \leq p < \infty$. We saw in Note 7.4.B that $l^\infty[a, b]$ is not separable.

Proposition 9.24. A compact metric space is separable.

Note. We now classify separable metric spaces in terms of the existence of a certain type of countable collection of open sets. In Chapter 11 (on topological spaces), we'll see in Section 11.1. Open Sets, Closed Sets, Bases, and Subbases that such a collection of sets form a countable "base" for topology on the metric space.

Proposition 9.25. A metric space X is separable if and only if there is a countable collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ of open subsets of X such that any open subset of X is the union of a subcollection of $\{\mathcal{O}_n\}_{n=1}^{\infty}$.

Note. As observed above, $\mathbb{R}$ is a metric space. The countable collection of open sets of real numbers, $\{\mathcal{O}_n\}_{n=1}^{\infty}$, guaranteed by Proposition 9.25 the the set of open intervals with rational coefficients. (Recall that a set of real numbers is open if and only if it is a countable disjoint union of open intervals; see Theorem 0.7, "Classification of Open Sets of Real Numbers, in Supplement. Essential Background for Real Analysis I.) Our next result concerns subspaces of separable metric spaces.

Proposition 9.26. Every subspace of a separable metric space is separable.

Note. The next theorem holds by Proposition 9.26 and Note 9.6.A.

Theorem 9.27. The following are separable metric spaces:

(i) Each nonempty closed subset of Euclidean space $\mathbb{R}^n$.

(ii) For E a measurable set of real numbers and $1 \leq p < \infty$, each nonempty subset of $L^p(E)$.

(iii) Each nonempty subset of $C[a, b]$.

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