Supplement. The Dirac Delta Function, A Cautionary Tale

Note. In the study of charge distributions in electricity and magnetism, when considering point charges it is common to introduce the “Dirac delta function,” $\delta(x)$. This function is defined to be extended real valued with the following properties:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) \, dx = 1.$$ 

However, this is very misleading! First, Riemann integrals do not address extended real valued functions. Second, such a function would violate Proposition 4.9 of Royden and Fitzpatrick’s Real Analysis, 4th Edition:

**Proposition 4.9.** Let $f$ be a nonnegative measurable (extended real valued) function on set $E$. Then the Lebesgue integral $\int_E f = 0$ if and only if $f = 0$ a.e. on $E$.

Note. In this supplement, we refer to the junior-level undergraduate text Foundations of Electromagnetic Theory, 3rd Edition by John Reitz, Frederick Milford, Robert Christy, Addison-Wesley (1979). We reveal some errors in it’s presentation of $\delta(x)$ and explore an alternative approach to “get the math right.”
Note. Reitz, Milford, and Christy introduce the Dirac delta function on pages 43 and 44.

Note. The motivation for $\delta(x)$ is given on page 43. On page 44, they comment that “...it is a legitimate mathematical object, which leads to no difficulties if one is cautious...”. In the footnote, they falsely state: “The Riemann integral of such a function is zero if it exists at all, but the integration can be handled by the more general Lebesgue integral.” Well, the Lebesgue integral does “handle” the integration, but it gives zero as the integral value.
**Definition.** Reitz, Milford, and Christy give more details in their Appendix IV, and the notation there is consistent with that used above in these notes (as opposed to a function of a vector variable as stated on their pages 43 and 44).

Note. In equations IV-2, IV-3, and IV-4 they write $\delta(x)$ as the limit of three different functions. In equation IV-2, they state $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi \varepsilon}} e^{-x^2/\varepsilon^2}$ (a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = \varepsilon/\sqrt{2}$). Since this is a normal distribution, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi \varepsilon}} e^{-x^2/\varepsilon^2} \, dx = 1$. As $\varepsilon \to 0$, the distribution gets very narrow with a large spike at $x = 0$. If we replace $\varepsilon$ with $1/n$ where $n \in \mathbb{N}$, then we can create a sequence of functions $f_n = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$ where $\lim_{n \to \infty} f_n(x) = \delta(x)$. 

The theorem gives the Fourier transform

$\hat{\delta}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) \, dx.$

For $f(x) = \delta(x)$, we find $\hat{f}(k) = 1/2\pi$ (as below), so that

$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk.$

Thus the Fourier transform of the delta function is a constant function, and conversely. Taking the real part of (IV-5) gives

$\delta(x) = \frac{1}{2\pi} \cos kx \, dk.$

Equations (IV-5) and (IV-6) may be considered as other representations of the delta function.

The mean-value theorem of calculus gives

$\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = F(0),
\int_{-\infty}^{\infty} \delta(x) \, dx = F(0),

$ so that as $f(x) \to \delta(x)$

$\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = F(0).$

In all of the above equations one can replace the variable $x$ by $(x - x_o)$; in particular,

$\int_{-\infty}^{\infty} f(x) \delta(x - x_o) \, dx = F(x_o).$

In any equation involving the delta function, the domain of integration can be reduced to any interval containing the point where the argument of the delta function vanishes.

An extension to three dimensions can be achieved by writing

$\delta(x) \, dx = \delta(x) \delta(y) \delta(z) \, dx \, dy \, dz.$

In Chapter 2 we found that

$\delta(x) = \nabla^2 \left( \frac{1}{|x|} \right).$
Note. Recall that from Royden and Fitzpatrick:

**Monotone Convergence Theorem.** Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on set \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.
\]

In the previous note, we have \( f_n = \frac{n}{\sqrt{\pi}} e^{-nx^2} \) and \( \lim_{n \to \infty} f_n(x) = \delta(x) \), but the convergence is not monotone increasing, so the Monotone Convergence Theorem does not apply to give that the integral \( \int_{-\infty}^{\infty} \delta(x) \, dx \) is 1. It is clear that Reitz, Milford, and Christy are trying to use the values of the integrals of the normal distributions to justify the claim that the integral of \( \delta(x) \) should be 1, but this is not the case.

Note. Recall that from Royden and Fitzpatrick:

**Fatou’s Lemma.** Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on set \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

\[
\int_E f \leq \liminf \left( \int_E f_n \right).
\]

Now we can apply Fatou’s Lemma to the sequence of normal distributions above, since the convergence to \( \delta(x) \) is pointwise and all functions here are nonnegative. But this simply allows us to conclude that

\[
\int_{-\infty}^{\infty} \delta(x) \, dx \leq \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-nx^2} \, dx = 1,
\]

still not an equality of 1, but merely an upper bound of 1.
Note. Reitz, Milford, and Christy’s “no difficulties if one is cautious” comment is correct, with the right kind of caution! In Analysis 2 (MATH 4227/5227) Riemann-Stieltjes integrals are introduced (see my online class notes on 6-3. The Riemann-Stieltjes Integral; in particular, see Theorem 6-26). One way to resolve the desired properties of the Dirac delta function, is to use Riemann-Stieltjes integrals. You should review these notes before reading the next note.

Note. Let

\[ g(x) = \begin{cases} 
0 & \text{for } x \in (-\infty, 0) \\
1 & \text{for } x \in [0, \infty). 
\end{cases} \]

Then the derivative of \( g \) is 0 if \( n \neq 0 \). The definition of limit gives that the derivative of \( g \) is \( \infty \) at 0. Also, we have the Riemann-Stieltjes integral

\[ \int_{-\infty}^{\infty} dg = \int_{-\infty}^{0} dg + (1) \left[ \lim_{x \downarrow 0} g(x) - \lim_{x \uparrow 0} g(x) \right] + \int_{0}^{\infty} dg = 0 + (1)[1] + 0 = 1. \]

So the derivative of \( g(x) \) has the values of \( \delta(x) \) given above and the Riemann-Stieltjes integral of \( f(x) = 1 \) with respect to \( g \) satisfies the integral property of \( \delta(x) \) given above. If \( f \) is a function defined on all of \( \mathbb{R} \), then we can use the Riemann-Stieltjes integral to determine the value of \( f \) at a specific point (say \( x = x_0 \)):

\[ \int_{-\infty}^{\infty} f(x) \, dg(x - x_0) = f(x_0) \left[ \lim_{x \downarrow x_0} g(x - x_0) - \lim_{x \uparrow x_0} g(x - x_0) \right] = f(x_0)[1] = f(x_0). \]

Note. Another, more sophisticated solution involves the Dirac measure concentrated at \( x_0 \), denoted \( \delta_{x_0} \). For a \( \sigma \)-algebra \( \mathcal{M} \) of subsets of a set \( X \) and a point \( x_0 \) belonging to \( X \), the measure of 1 is assigned to a set in \( \mathcal{M} \) that contains \( x_0 \), and a measure of 0 is assigned to a set that does not contain \( x_0 \). We then have for measurable function \( f \) on \( X \) that \( \int_X f \, d\delta_{x_0} = f(x_0) \), as desired. This is explored in Problem 18.26 (in 18.2. Integration of Nonnegative Measurable Functions) and Problem 18.27(ii) (in 18.3. Integration of General Measurable Functions).
Note. The Dirac delta function is named for Paul A. M. Dirac (August 8, 1902–October 20, 1984). Another approach is to treat \( \delta(x) \) not as a function, but as a distribution (or a generalized function). It is therefore more accurately called the “Dirac delta distribution.” A classical work on the theory of distributions is I. M. Gel’fand and G. E. Shilov’s *Generalized Functions*, Volumes 15, Academic Press, (1966-1968).

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