Section 54. The Fundamental Group of the Circle

Note. In this section, we tie together the idea of a covering space from the previous section with the idea of the fundamental group from Section 52. We also show that the fundamental group of the circle $S^1$ is isomorphic to $\mathbb{Z}$.

Definition. Let $p : E \to B$ be a map between space $E$ and space $B$. If $f$ is a continuous mapping of some space $X$ into $B$, then a lifting of $f$ is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

Note. The diagram for these maps is:

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\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (E) at (2,0) {$E$};
  \node (B) at (2,-2) {$B$};
  \draw[->] (X) -- (E) node [above] {$\tilde{f}$};
  \draw[->] (X) -- (B) node [left] {$f$};
  \draw[->] (E) -- (B) node [right] {$p$};
\end{tikzpicture}
```

Example 54.1. Consider the covering $p : \mathbb{R} \to S^1$ of Theorem 53.1. The path $f : [0,1] \to S^1$ given by $f(s) = (\cos(\pi s), \sin(\pi s))$ (a path along $S^1$ from $b_0 = (1,0)$ to $(-1,0)$) lifts to the path $\tilde{f}(s) = s/2$ in $\mathbb{R}$ beginning at 0 and ending at $1/2$, because $p \circ \tilde{f}$ maps $[0,1]$ to the upper half of $S^1$ and $f$ maps $[0,1]$ to this same upper half of $S_1$: 
The path \( g(s) = (\cos(\pi s), -\sin(\pi s)) \) lifts to the path \( \tilde{g}(s) = -s/2 \) beginning at 0 and ending at \(-1/2\) (as above, but now involving the lower half of \( S^1 \)). The path \( h(s) = (\cos(4\pi s), \sin(4\pi s)) \) lifts to the path \( \tilde{h}(s) = 2s \) beginning at 0 and ending at 2:

**Note.** In the next two results, we show that for a covering space, (1) paths can be lifted, and (2) path homotopies can be lifted.

**Lemma 54.1.** Let \( p : E \to B \) be a covering map, and let \( p(e_0) = b_0 \). Any path \( f : [0, 1] \to B \) beginning at \( b_0 \) has a unique lifting to a path \( f \) in \( E \) beginning at \( e_0 \).
Lemma 54.2. Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous with $F(0,0) = b_0$. There is a unique lifting of $F$ to a continuous map $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If $F$ is a path homotopy, then $\tilde{F}$ is a path homotopy.

Note. The next result shows that homotopic paths are lifted to homotopic paths.

Theorem 54.3. Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Let $f$ and $g$ be two paths in $B$ from $b_0$ to $b_1$. Let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in $E$ beginning at $e_0$. If $f$ and $g$ are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of $E$ and are path homotopic.

Note. We define a mapping which will be useful in determining the fundamental group of the circle $S^1$.

Definition. Let $p : E \to B$ be a covering map. Let $b_0 \in B$. Choose $e_0$ so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B,b_0)$, let $\tilde{f}$ be the lifting of $f$ to a path in $E$ that begins at $e_0$ (we know by Lemma 54.1 that $\tilde{f}$ is well-defined in terms of any $f \in [f]$). Let $\varphi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then

$$\varphi : \pi_1(B,b_0) \to p^{-1}(b_0).$$

is the lifting correspondence derived from the covering map $p$. 
Theorem 54.4. Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. If $E$ is path connected, then the lifting correspondence

$$\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective (onto). If $E$ is simply connected, it is bijective.

Theorem 54.5. The fundamental group of $S^1$ is isomorphic to the additive group of integers, $\mathbb{Z}$.

Note. The previous result justifies our intuitive idea that the fundamental group of $S^1$ is generated by starting at $(1,0)$ and creating loops that wrap around $S^1$ a positive integer number of times (counterclockwise) and loops that wrap around $S^1$ a negative integer number of times (clockwise). The intuitive pasting together of loops corresponds to the binary operation in the fundamental group.

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