Chapter 1. Four-Dimensional Vector Spaces and Linear Mappings

1.2. Lorentz Mappings of $\mathbb{V}_4$

Definition. A *linear mapping* $\mathbf{L} : \mathbb{V}_4 \to \mathbf{V}_4$ is a mapping such that

$$\mathbf{L}(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda\mathbf{L}(\mathbf{a}) + \mu\mathbf{L}(\mathbf{b})$$

for all $\lambda, \mu \in \mathbb{R}$ and all $\mathbf{a}, \mathbf{b} \in \mathbb{V}_4$.

Definition. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis for $\mathbb{V}_4$, then define $\hat{\mathbf{e}}_j = \mathbf{L}(\mathbf{e}_j) = l^i_j\mathbf{e}_i$. The matrix $L = [l^i_j]$ is the *matrix representation* of linear mapping $\mathbf{L}$ relative to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ of $\mathbb{V}_4$.

Note. For $\mathbf{u} = u^i e_i \in \mathbb{V}_4$, we have $\mathbf{L}(\mathbf{u}) = \mathbf{L}(u^i e_i) = u^i\mathbf{L}(\mathbf{e}_i) = u^i\hat{\mathbf{e}}_i = u^j\hat{\mathbf{e}}_j = u^j l^i_j e_i$.

Definition. A linear mapping $\mathbf{L}$ with $\det[l^i_j] \neq 0$ is *invertible*.

Theorem 1.2.1. Let $\hat{\mathbf{e}}_i = \mathbf{l}(\mathbf{e}_i)$. The set of vectors $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ is also a basis for $\mathbb{V}_4$ if and only if $\det[l^i_j] \neq 0$.

Proof. ...

Example. A linear mapping which leaves $\mathbf{e}_3$ and $\mathbf{e}_4$ coordinates fixed and produces a $\frac{\pi}{4}$ rotation in the $\mathbf{e}_1\mathbf{e}_2$-plane:

$$\begin{aligned}
\hat{\mathbf{e}}_1 = \mathbf{L}(\mathbf{e}_1) &= \cos(\pi/4)\mathbf{e}_1 + \sin(\pi/4)\mathbf{e}_2 = l^i_1\mathbf{e}_i \\
\hat{\mathbf{e}}_2 = \mathbf{L}(\mathbf{e}_2) &= \sin(\pi/4)\mathbf{e}_1 + \cos(\pi/4)\mathbf{e}_2 = l^i_2\mathbf{e}_i \\
\hat{\mathbf{e}}_3 = \mathbf{L}(\mathbf{e}_3) &= l^i_3\mathbf{e}_i \\
\hat{\mathbf{e}}_4 = \mathbf{L}(\mathbf{e}_4) &= l^i_4\mathbf{e}_i.
\end{aligned}$$

Then the matrix representation of $\mathbf{L}$ is

$$L = [l^i_j] = \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) & 0 & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Of course $\mathbf{L}$ is invertible (in fact $\det(\mathbf{L}) = 1$). The inverse of $\mathbf{L}$ rotates the $\mathbf{e}_1\mathbf{e}_2$-plane throughan angle of $-\pi/4$ and has matrix representation

$$\begin{aligned}
L^{-1} &= \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) & 0 & 0 \\ \sin(-\pi/4) & \cos(-\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) & 0 & 0 \\ -\sin(\pi/4) & \cos(\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

The *Kronecker delta* is defined as $\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Theorem 1.2.2 Suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ are bases for $\mathbb{V}_4$ and that $\hat{\mathbf{e}}_j = \mathbf{L}(\mathbf{e}_j) = l^i_j \mathbf{e}_i$. Sice $\det[l^i_j] \neq 0$ by Theorem 1.2.1, $\mathbf{L}^{-1}$ exists, say $\mathbf{L}^{-1} = \mathbf{A} = [a^i_k]$. For $\mathbf{u} = u^k \mathbf{e}_k = \hat{u}^j \hat{\mathbf{e}}_j$ we have the components of $\mathbf{u}$ in terms of $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ satisfy $\hat{u}^i = a^i_k u^k$.

Proof. We have $\mathbf{u} = u^k \mathbf{e}_k = \hat{u}^j \hat{\mathbf{e}}_j = \hat{u}_j \mathbf{L}(\mathbf{e}_j) = \hat{u}^j l^k_j \mathbf{e}_k = (l^k_j \hat{u}^j)\mathbf{e}_k$. So $(u^k - l^k_j \hat{u}^j)\mathbf{e}_k = \mathbf{0}$. Therefore $u^k = l^k_j \hat{u}^j$, or $a^i_k u^k = a^i_k l^k_j \hat{u}^j = \delta^i_j \hat{u}^j = \hat{u}^i$. ∎

Note. Theorem 1.2.2 allows us to transform the components of a given vector from one basis to another using matrices (this is the "Change of Basis Theorem" of linear algebra). So far in this section we have ol presented results normally encountered in linear algebra. We now deviate from this approach and concentrate more specifically on $\mathbb{V}_4$.

Definition. A linear mapping $\mathbf{L} : \mathbb{V}_4 \to \mathbb{V}_4$ which satisfies $\mathbf{L}(\mathbf{a}) \cdot \mathbf{L}(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{V}_4$ is a *Lorentz mapping*. Then corresponding matrix $L = [l^i_j]$ relative to an M-orthonormal basis is a *Lorentz matrix*.

Note. A Lorentz mapping preserves inner products and hence maps timelike/spacelike/null vectors to timelike/spacelike/null vectors respectively.

Theorem 1.2.3 A Lorentz matrix L satisfies $L^T DL = D$ where L^T is the transpose of L and D is the matrix of the Lorentz metrix.

Proof. By the definition of a Lorentz mapping

$$\mathbf{L}(\mathbf{e}_i) \cdots \mathbf{L}(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = d_{ij} \qquad (1.2.9)$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is an M-orthonormal basis. With $\mathbf{L}(\mathbf{e}_j) = l_i^k \mathbf{e}_k$, equation (1.2.9) yields $(l_i^k \mathbf{e}_k) \cdot (l_j^m \mathbf{e}_m) = d_{ij}$, so

$$\begin{aligned} L^T DL &= [l_i^k d_{km} l_j^m] = [l_i^k l_j^m d_{km}] = [l_i^k l_j^m (\mathbf{e}_k \cdot \mathbf{e}_m)] \\ &= [(l_i^k \mathbf{e}_k) \cdot (l_j^m \mathbf{e}_m)] = [d_{ij}] = D. \end{aligned}$$

∎

Note. Theorem 1.2.3 is primarliy of interest because it implies the following corollary.

Corollary 1.2.1 The determinant of a Lorentz matrix L satisfies $\det L = \det[l_j^i] = \pm 1$.

Proof. From Theorem 1.2.3, we have $\det(L^T DL) = \det(D)$ or $\det(L^T)\det(D)\det$ $\det(D)$. Since $\det(L^T) = \det L$ we have $\det(L)^2 = 1$ and the result follows.

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Definition. A Lorentz mapping $\mathbf{L}$ with $\det L = 1$ is a *proper Lorentz mapping.* A Lorentz mapping $\mathbf{L}$ with $\det L = -1$ is an *improper Lorentz* *mapping.* A Lorentz mapping $\mathbf{L}$ such that $l^4_4 > 0$ when represented in an M-orthonormal basis is called an *orthochronous Lorentz mapping.*

Example. The *space reflection* $\mathbf{P}$ is given by the mappings $\mathbf{P}(\mathbf{e}_\alpha) = -\mathbf{e}_\alpha$ and $\mathbf{P}(\mathbf{e}_4) = \mathbf{e}_4)$. The matrix representation of $\mathbf{P}$ is

$$P = [p^i_j] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\mathbf{P}$ is an improper orthochronous Lorentz mapping and $\mathbf{P}^{-1} = \mathbf{P}$.

Example. The *time reversal* $\mathbf{T}$ is given by $\mathbf{T}(\mathbf{e}_\alpha) = \mathbf{e}_\alpha$ and $\mathbf{T}(\mathbf{e}_4) = -\mathbf{e}_4$. The matrix represntation of $\mathbf{T}$ is

$$T = [t^i_j] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

$\mathbf{T}$ is an improper nonorthochronous Lorentz mapping.

Example. We can generalize an earlier example by rotating through an angle θ in the $\mathbf{e}_1\mathbf{e}_2$-plane and fixing $\mathbf{e}_3$ and $\mathbf{e}_4$. We have such a Lorentz transformation **L** satisfying

$$L = [l^i_j] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then **L** is proper and orthochronous.

Example. A transformation that will be important later is the *boost mapping*. It represents a transformation from one frame of reference to another frame which moves uniformly with respect to the first. First, let $|v| < 1$ (we'll interpret v as a velocity measured in units of the speed of light c, here taken to be $c = 1$) and $\beta = (1 - v^2)^{-1/2}$. We want

$$\begin{aligned} \mathbf{L}(\mathbf{e}_1) &= \beta(\mathbf{e}_1 + v\mathbf{e}_4) = \hat{\mathbf{e}}_1 \\ \mathbf{l}(\mathbf{e}_2) &= \mathbf{e}_2 = \hat{\mathbf{e}}_2 \\ \mathbf{L}(\mathbf{e}_3) &= \mathbf{e}_3 = \hat{\mathbf{e}}_3 \\ \mathbf{L}(\mathbf{e}_4) &= \beta(v\mathbf{e}_1 + \mathbf{e}_4) = \hat{\mathbf{e}}_4. \end{aligned}$$

Then

$$L = [l^i_j] = \begin{bmatrix} \beta & 0 & 0 & v\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\beta & 0 & 0 & \beta \end{bmatrix}.$$

We can verify that **L** is proper and orthochronous. It is common to represent this situation in the $\mathbf{e}_1\mathbf{e}_4$-plane and define ϕ where $\tan\phi = v$ (and so $|\phi| < \pi/4$). We can draw the axes as follows:

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This looks odd since $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_4$ are still M-orthogonal, but remember that M-orthogonality is nonintuitive when it involves vectors with a timelike component. Notice also that as $v \to c = 1$ then $\phi \to \pi/4$ and $\hat{\mathbf{e}}_1 \to \hat{\mathbf{e}}_4$. This means the gap between $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_4$ will decrease — the result will be length contraction and time dilation, as we will see later.