Chapter 1. Four-Dimensional Vector Spaces and Linear Mappings

1.3. The Minkowski Tensor

Definition. A *Minkowski covector* $\tilde{\mathbf{w}}$ (or a *covariant vector*) is a linear mapping from $\mathbb{V}_4$ to $\mathbb{R}$. The *components* of $\tilde{\mathbf{w}}$ are defined as $w_i = \tilde{\mathbf{w}}(\mathbf{e}_i)$ (notice the use of subscripts for covectors, where we have used superscripts for "regular" vectors…sometimes called *contravariant vectors*).

Theorem 1.3.1 The components w_i of a covector $\tilde{\mathbf{w}}$ under a Lorentz mapping $\mathbf{L}$ where $L = [l_j^k]$ transforms as $\hat{w}_j = l_j^k w_k$.

Proof. We have $\hat{\mathbf{e}}_j = \mathbf{L}(\mathbf{e}_j) = l_j^k \mathbf{e}_k$, so

$$\begin{aligned} \hat{w}_j &= \tilde{\mathbf{w}}(\mathbf{e}_j) \text{ by definition} \\ &= \tilde{\mathbf{w}}(l_j^k \mathbf{e}_k) \\ &= l_j^k \tilde{\mathbf{w}}(\mathbf{e}_k) \text{ since } \tilde{\mathbf{w}} \text{ is linear} \\ &= l_j^k w_k. \end{aligned}$$

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Note. As we will see, different frames of reference yield different measurements of, for example, lengths and time increments. These quantities are *relative*, whereas the measurement of the speed of light is an absolute (that is, it is an invariant from one frame of reference to another). We use the Lorentz mappings to transform quantities from one frame of reference to another, therefore we are interested in quantities which are invariant under Lorentz mappings.

Corollary 1.3.1 Let u^i, w_i be components of a vector $\mathbf{u}$ and a covector $\tilde{\mathbf{w}}$, respectively. The sum $u^i w_i$ remains invariant under a Lorentz mapping.

Proof. Let $\hat{u}^i$ and $\hat{w}_i$ be the components of $\mathbf{L}(\mathbf{u})$ and $\mathbf{L}\tilde{\mathbf{w}})$, respectively. Then

$$\begin{aligned}
\hat{u}^i \hat{w}_i &= (a^i_k u^k)(l^m_i w_m) \text{ by Theorem 1.2.2 and Theorem 1.3.1} \\
&= (l^m_i a^i_k) u^k w_m \\
&= \delta^m_k u^k w_m \text{ since } L^{-1} = [l^m_i]^{-1} = [a^i_k] = A \\
&= u^k w_k.
\end{aligned}$$

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Definition. The number $u^i w_i$ where u^i and w_i are components of a vector $\mathbf{u}$ and a covector $\tilde{\mathbf{w}}$, respectively, is a *Lorentz* (or *Minkowski*) *scalar*.

Note. Corollary 1.3.1 implies that Lorentz scalars are invariant under Lorentz mappings.

Definition. A *bilinear mapping* $\mathbf{T} : \mathbb{V}_4 \times \mathbb{V}_4 \to \mathbb{R}$ satisfies

$$\begin{aligned} \mathbf{T}(\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{c}) &= \lambda\mathbf{T}(\mathbf{a}, \mathbf{c}) + \mu\mathbf{T}(\mathbf{b}, \mathbf{c}) \\ \mathbf{T}(\mathbf{a}, \lambda\mathbf{b} + \mu\mathbf{c}, \mathbf{c}) &= \lambda\mathbf{T}(\mathbf{a}, \mathbf{b}) + \mu\mathbf{T}(\mathbf{a}, \mathbf{c}) \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}$ and all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_4$. $\mathbf{T}$ is called a *second-order covariant Minkowski tensor* and define its components as $\tau_{ij} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j)$.

Note. The components τ_{ij} of a second order covariant tensor under Lorentz mapping $\mathbf{L}$ where $L = [l_i^m]$ transforms as $\hat{\tau}_{ij} = l_i^m l_j^k \tau_{mk}$. This is because:

$$\begin{aligned} \hat{\tau}_{ij} &= \mathbf{T}(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j) = \mathbf{T})\mathbf{L}(\mathbf{e}_i), \mathbf{L}(\mathbf{e}_j)) = \mathbf{T}(l_i^k \mathbf{e}_m, l_j^k \mathbf{e}_k) \\ &= l_i^m l_j^k \mathbf{T}(\mathbf{e}_m, \mathbf{e}_k) = l_i^m l_j^k \tau_{mk}. \end{aligned}$$

Definition. A *second-order contravariant Minkowski tensor* is a bilinear mapping $\mathbf{T}$ with components $\tau^{ij} = (\mathbf{e}_i, \mathbf{e}_j)$ which transforms under Lorentz mapping $\mathbf{L}$ as $\hat{\tau}^{ij} = a^i_m a^i_k \tau^{mk}$ where $L = [l^j_i]$ and $A = L^{-1} = a^j_i]$.

Definition. A *second-order mixed Minkowski tensor* is a bilinear mapping $\mathbf{T}$ with components $\tau^i_j = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j)$ which transforms under Lorentz mapping $\mathbf{L}$ as $\hat{\tau}^i_j = a^i_m l^k_j \tau^m_k$ where $L = [l^j_i]$ and $A = L^{-1} = [a^j_i]$.

Note. OK, here's the deal with the contravariant/covariant thing for tensors:

1. A tensor is a multilinear mapping (the number of linear parts determines the *order*).
2. The components of the tensor are determined by applying the tensor to basis elements (for an order n tensor, there are 4^n components).
3. The covariant tensors have components which transform under $\mathbf{L}$ using l^j_i.
4. The contravariant tensors have components which transform under $\mathbf{L}$ using a^j_i where $L^{-1} = A = [a^j_i]$.

Definition. A second-order covariant tensor is *antisymmetric* if its components α_{ij} satisfy $\alpha_{ij} = -\alpha_{ji}$. A second-order covariant tensor if *symmetric* if the components σ_{ij} satisfy $\sigma_{ij} = \sigma_{ji}$.

Note. The number of "linearly independent" components of an antisymmetric Minkowski tensor of order two is 6. Since $\alpha_{ij} = -\alpha_{ji}$, diagonal entries must be 0, and the 6 entries above the diagonal determine the 6 entries below the diagonal. In a symmetric Minkowski tensor, we cannot determine the diagonal entries and so there are $6 + 4 = 10$ independent entries.