Equation Sheet for Differential Geometry (MATH 5310)

Chapter 1. Surfaces and the Concept of Curvature

Curves

Curve: $\vec{\alpha}(t)$ or $\vec{\alpha}(s)$ where s is arclength

Unit Tangent Vector: $\vec{T}(t) = \dfrac{\vec{\alpha}(t)}{\|\vec{\alpha}'(t)\|}$ or $\vec{T}(s) = \vec{\alpha}'(s)$

Principal Normal Vector: $\vec{N}(s) = \dfrac{\vec{T}'(s)}{\|\vec{T}'(s)\|}$

Curvature: $k(s) = \|\vec{T}(s)\| = \|\vec{\alpha}''(s)\|$

Center of Curvature: $\vec{c}(s) = \vec{\alpha}(s) + \dfrac{\vec{N}(s)}{k(s)}$

The Osculating Plane is determined by: $\vec{T}$ and $\vec{N}$

Normal Vector for the Osculating Plane: $\vec{B} = \vec{T} \times \vec{N}$ (the "Binormal Vector")

Surfaces

Surface: $\vec{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ or $\vec{X}(u^1,u^2) = (x(u^1,u^2), y(u^1,u^2), z(u^1,u^2))$

$\vec{X}_1(u,v) = \dfrac{\partial \vec{X}}{\partial u}$ and $\vec{X}_2(u,v) = \dfrac{\partial \vec{X}}{\partial v}$, or $\vec{X}_i(u^1,u^2) = \dfrac{\partial \vec{X}}{\partial u^i}$

The metric coefficients: $g_{ij} = \vec{X}_i \cdot \vec{X}_j$.

Gauss's Notation: $E = g_{11}$, $F = g_{12} = g_{21}$, $G = g_{22}$

The Metric Form:

$$\begin{aligned}(ds)^2 &= E(du)^2 + 2F(du\,dv) + G(dv)^2 \\ &= g_{11}(du)^2 + g_{12}(du\,dv) + g_{21}(dv\,du) + g_{22}(dv)^2\end{aligned}$$

Matrix of the First Fundamental Form: $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$; $g = \det(g_{ij})$

Unit Normal Vector: $\vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}$

Second Partials: $\vec{X}_{ij} = \dfrac{\partial^2 \vec{X}}{\partial u^i\,\partial u^j}$

Formulae of Gauss: $\vec{X}_{ij} = \Gamma^r_{ij}\vec{X}_r + L_{ij}\vec{U}$ (17)

Components of Acceleration, $\vec{\alpha}''(s)$:

$$\vec{\alpha}'' = (u^{r\,\prime\prime} + \Gamma^r_{ij}u^{i\,\prime}u^{j\,\prime}\vec{X}_r + (L_{ij}u^{i\,\prime}u^{j\,\prime})\vec{U} \qquad (18)$$

$$\vec{\alpha}''_{\text{nor}} = (u^{r\,\prime\prime} + \Gamma^r_{ij}u^{i\,\prime}u^{j\,\prime}\vec{X}_r \qquad (19a)$$

$$\vec{\alpha}''_{\text{nor}} = (L_{ij}u^{i\,\prime}u^{j\,\prime})\vec{U} \qquad (19b)$$

Second Fundamental Form: $\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ and $L = \det(L_{ij})$ where

$$L_{ij} = \vec{X}_{ij} \cdot \vec{U} \qquad (20)$$

Normal Curvature in the Direction $\vec{v}$: $k_v(\vec{v}) = L_{ij}v^i v^j$ (21)

Gauss Curvature at Point $\vec{P}$: $K(P) = k_1 k_2$ where $k_1 = \max k_n(\vec{v})$ and $k_2 = \min k_n(\vec{v})$

Calculating Gauss Curvature using Theorem I-5: $K(\vec{P}) = L/g$

Definition and computation of L^i_j: $L^i_j = L_{jk}g^{ki}$ (27)

$$L^i_j g_{im} = L_{jm} \qquad (27')$$

Equations of Weingarten: $\vec{U}_j = -L^i_j \vec{X}_i$ (28)

Geodesic Curvature of $\vec{\alpha}$ is k_g where: $\vec{\alpha}''_{\text{tan}} = k_g(\vec{U} \times \vec{\alpha}')$ (30)

Computation of Geodesic Curvature: $k_g = \vec{U} \cdot \vec{\alpha}' \times \vec{\alpha}''$ (31)

If $\vec{\alpha}$ is a geodesic then: $u^{r\prime\prime} + \Gamma^r_{ij} u^{i\prime} u^{j\prime} = 0$ and $\vec{U} \cdot \vec{\alpha}' \times \vec{\alpha}'' = 0$ (32a and 32b)

Christoffel Symbols of the First Kind: $\Gamma_{ijk} = \Gamma^r_{ij} g_{rk}$ (33)

Computation of Christoffel Symbols:

$$\Gamma_{ijk} = \vec{X}_{ij} \cdot \vec{X}_k \qquad (34)$$

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \qquad (36)$$

$$\Gamma^r_{ij} = \frac{1}{2} g^{kr}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \qquad (37)$$

Riemann-Christoffel Curvature Tensor:

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial u^j} - \frac{\partial \Gamma^h_{ij}}{\partial u^k} + \Gamma^r_{ik}\Gamma^h_{rj} - \Gamma^r_{ij}\Gamma^h_{rk} \qquad (51)$$

Computation of Curvature of Surface $\vec{X}(u,v)$ (from Corollary 1.8.A):

$$K = \frac{1}{g}\left[F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} + (\Gamma^h_{12}\Gamma^r_{12} - \Gamma^h_{22}\Gamma^r_{11})g_{rh}\right]$$

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