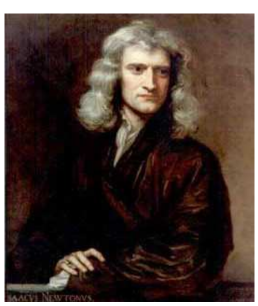
1.1 Curves

Note. The historical notes in this section are based on Morris Kline's *Mathematical Thought From Ancient to Modern Times*, Volume 2, Oxford University Press (1972), "Analytic and Differential Geometry in the Eighteenth Century" (Chapter 23). Page numbers in the historical notes refer to this reference. Images are from the MacTutor History of Mathematics archive.

Note. We define the curvature of a path (or "curve) in 3-dimensional space in this section. We inspire our definitions by appealing to physical intuition based on velocities and accelerations. Therefore, there is a need to employ the techniques of calculus. The term "differential geometry" was first used by Luigi Bianchi (1856–1928) in 1894. See page 554.

Note. Isaac Newton (1642–1727) studied the curvature of plane curves (that is, curves in a plane). His work was published in *Geometricia Analytica* (in 1736 though his work seems to date to 1671). He does so by introducing a center of curvature as the limiting point of the intersection of normal curves near a given point. He defines the curvature at the point as reciprocal of the radius of the resulting "osculating circle" (a term used by Gottfried Leibniz [1646–1716] in a paper in 1686). Newton calculated the curvature of several curves and as a result duplicated some earlier work of Christiaan Huygens (1629–1695). See pages 556 and 557.



Isaac Newton

Gottfried Leibniz

Note. The study of curves in 3-dimensional space (as opposed to those restricted to a plane as discussed above) was extensively studied by Alexis-Claude Clairaut (1713–1765). He gave expressions for the arc length of a space curve and the quadrature of certain areas on surfaces. However, very little had been done in the theory of surfaces by 1750. See page 557.


Alexis Clairaut


Leonhard Euler

Note. The next major step in the study of curves in 3-dimensional space is due to Leonhard Euler (1707–1783). He included much of this work in his *Mechanica* (published in 1736) and presented more results in his *Theoria Motus Corporum Solidorum seu Rigidorum* (published in 1765). In this second book Euler derived the formulas for the radial and normal components of acceleration in polar coordinates for a plane curve $f(t)$. These are given in Calculus 3 and used in the proof of Kepler's First Law; see my online notes for "Velocity and Acceleration in Polar Coordinates" at `faculty.etsu.edu/gardnerr/2110/notes-12e/c13s6.pdf`. Euler started to explore curves in 3-dimensional space in 1774 and presented a paper with his results in 1775. See page 558.

Note. As we do in this section, Euler represented space curves by the parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is arc length. He defines the *osculating plane* associated with a curve in space, as we will do (though the term "osculating plane" is due to Johann Bernoulli, 1667-1748). See pages 558 and 559.

Note. Alexis Clairaut introduced the idea that a curve in 3-dimensional space has two curvatures, which we shall call "curvature" and "torsion." Torsion was formulated explicitly and analytically by Michel-Ange Lancret (1774–1807). Lancret also called attention to three principal directions associated with each point on a curve in 3-dimensional space in his *Mémoires de Mathématique et de Physique Présentés à l'Académie Royal des Sciences, par Divers Sçavans, et Lus dans ses Assemblées*, **1** (1806), 416–454. These directions correspond to our unit tangent vector $\vec{T}$, principal normal vector $\vec{N}$, and binormal vector $\vec{B}$.

Michel-Ange Lancret
(from Wikipedia)

Augustin Louis Cauchy

Note. Augustin Louis Cauchy (1789–1857) in his *Leçons sur les applications du calcul infinitésimal à la géométrie* (published in 1826) clarifies much of the theory of curves in 3-dimensions. He also cleans up many of the conceptual problems with calculus involving the vague concept of infinitesimals and differentials. In connection with curves, he points out that when one writes $ds^2 = dx^2 + dy^2 + dz^2$ one should mean

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2.$$

Cauchy's development of the geometry of curves is practically modern. See page 560.

Note. We now turn our attention to Faber's text. In this section, we encounter most of the ideas mentioned above in the historical survey. We motivate the ideas and definitions by appealing to physical intuition based on the position, velocity, and acceleration experienced by a particle as it follows a curve in 3-dimensional space $\mathbb{R}^3 = E^3$.

Definition. A *curve* in E^3 is a vector valued function of the parameter t:

$$\vec{\alpha}(t) = (x(t), y(t), z(t)).$$

Note. We assume the functions $x(t)$, $y(t)$, and $z(t)$ have continuous second derivatives.

Definition. The *derivative vector* of curve $\vec{\alpha}$ is

$$\vec{\alpha}'(t) = (x'(t), y'(t), z'(t)).$$

If $\vec{\alpha}(t)$ is the position of a particle at time t, then $\vec{\alpha}'(t)$ is the *velocity vector* of the particle and $\vec{\alpha}''(t)$ is the *acceleration vector* of the particle. The *speed* of the particle is the scalar function $\|\vec{\alpha}'(t)\|$.

Note. According to Newton's Second Law of motion, the force acting on a particle of mass m and position $\vec{\alpha}(t)$ is $\vec{F}(t) = m\vec{\alpha}''(t)$.

Definition. The *length* (or *arclength*) of the curve $\vec{\alpha}(t)$ for $t \in [a, b]$ is

$$S = \int_a^b \|\vec{\alpha}'(t)\| dt.$$

Note. If $\vec{\beta}(t)$ is a curve for $t \in [a,b]$ where $\vec{\beta}'(t)$ exists and is nonzero, then $\vec{\beta}$ can be written as a function of arclength (which we will denote $\vec{\alpha}(s)$) as follows. First,

$$S(t) = \int_a^t \|\vec{\beta}'(t)\| dt$$

(that is, $S(t)$ is an antiderivative of speed which satisfies $S(a) = 0$). Therefore S is a one to one function and S^{-1} exists. S^{-1} gives the time at which the particle has traveled along $\vec{\beta}(t)$ a (gross) distance s. So we denote this as $t = S^{-1}(s)$. Second, we make the substitution for t:

$$\vec{\beta}(t) = \vec{\beta}(S^{-1}(s)) \equiv \vec{\alpha}(s).$$

However, it may be algebraically impossible to calculate $t = S^{-1}(s)$ (see page 11, number 5).

Recall. If f is differentiable on an interval I and f' is nonzero on I, then f^{-1} exists (i.e. f is one-to-one on I) on $f(I)$ and f^{-1} is differentiable on I. In addition,

$$\left(\frac{df^{-1}}{dx}\right)\bigg|_{x=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)\Big|_{x=a}}$$

or $f^{-1\prime}(f(a)) = \dfrac{1}{f'(a)}$.

Note. If $\vec{\beta}(t)$ is parameterized as $\vec{\alpha}(s)$ as above, then

$$\vec{\beta}(t) = \vec{\beta}(S^{-1}(s)) = \vec{\alpha}(s)$$

and

$$\frac{d\vec{\alpha}}{ds} = \frac{d\vec{\beta}}{dS^{-1}}\frac{dS^{-1}}{ds} = \vec{\beta}'(S^{-1}(s))\frac{1}{S'(S^{-1}(s))} = \vec{\beta}'(t)\frac{1}{S'(t)} = \frac{\vec{\beta}'(t)}{\|\vec{\beta}'(t)\|}.$$

Notice $\dfrac{d\vec{\alpha}}{ds} = \vec{\alpha}'(s)$ is a unit vector in the direction of the velocity vector of $\vec{\beta}(t)$.

Definition. If $\vec{\alpha}(s)$ is a curve parameterized in terms of arclength s, then the *unit tangent vector* of $\vec{\alpha}(s)$ is $\vec{\alpha}'(s) = \vec{T}(s)$. ($\vec{\alpha}(s)$ is called a *unit speed curve* since $\|\vec{\alpha}'(s)\| = 1$.)

Example 3 (page 6). Consider the circular helix

$$\vec{\beta}(t) = (a\cos t, a\sin t, bt)$$

(see Figure I-3, page 6). Parameterize $\vec{\beta}(t)$ in terms of arclength $\vec{\alpha}(s)$ and calculate $\vec{T}(s)$.

Solution. We have $\vec{\beta}'(t) = (-a\sin t, a\cos t, b)$. With $S(t)$ the total arclength travelled by a particle along the helix at time t, we have

$$S'(t) = \|\vec{\beta}'(t)\| = \sqrt{a^2+b^2}.$$

Therefore, $S(t) = t\sqrt{a^2+b^2}$ (taking $S(0) = 0$). Hence

$$t = S^{-1}(s) = \frac{s}{\sqrt{a^2+b^2}}$$

and

$$\vec{\alpha}(s) = \vec{\beta}(t) = \vec{\beta}(S^{-1}(s)) = \vec{\beta}\left(\frac{s}{\sqrt{a^2+b^2}}\right)$$
$$= \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right).$$

Also,

$$\vec{T}(s) = \vec{\alpha}'(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\right).$$

Notice that

$$\vec{T} = \frac{\vec{\beta}'(t)}{\|\vec{\beta}'(t)\|} = \frac{\vec{\beta}'(S^{-1}(s))}{\|\vec{\beta}'(S^{-1}(s))\|}.$$

Note. $\vec{T}(s)$ always has unit length. The only way $\vec{T}(s)$ can change is in direction. Notice that this corresponds to a change in the direction of travel of a particle along the path $\vec{\alpha}(s)$. Since $\vec{T}(s)\cdot\vec{T}(s) = 1$, we have $\vec{T}'(s)\cdot\vec{T}(s) = 0$ (by the product rule) and so $\vec{T}'(s) = \vec{\alpha}''(s)$ is orthogonal to $\vec{T}(s) = \vec{\alpha}'(s)$.

Definition. The *curvature* of $\vec{\alpha}(s)$ (denoted $k(s)$) is

$$k(s) = \|\vec{T}'(s)\| = \|\vec{\alpha}''(s)\|.$$

If $\vec{T}'(s) \neq \vec{0}$ (and therefore curvature is nonzero) then the unit vector in the direction of $\vec{T}'(s)$ is the *principal normal vector*, denoted $\vec{N}(s)$.

Notice.

$$\vec{N}(s) = \frac{\vec{T}'(s)}{\|\vec{T}'(s)\|} = \frac{\vec{T}'(s)}{k(s)}.$$

Example 3, page 6 (cont.). Calculate the curvature $k(s)$ and principal normal vector $\vec{N}(s)$ for the helix

$$\vec{\beta}(t) = (a\cos t, a\sin t, bt).$$

Solution. From above, we have

$$\vec{T}'(s) = \vec{\alpha}''(s) = \frac{-1}{a^2+b^2}\left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$$

and so $k(s) = \|\vec{T}'(s)\| = \dfrac{|a|}{a^2+b^2}$ (a constant). Now

$$\vec{N}(s) = \frac{\vec{T}'(s)}{k(s)} = -\left(\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right) \quad \text{if } a > 0.$$

Notice that in terms of t,

$$\vec{N}(t) = -(\cos t, \sin t, 0).$$

That is, $\vec{N}(t)$ is a vector that points from the particle at $\vec{\beta}(t) = (a \cos t, a \sin t, bt)$ back to the $z-$axis (that is, $\vec{N}(t)$ is a unit vector from $\vec{\beta}(t)$ to $(0, 0, bt)$).

Note. If we take $b = 0$ in Example 3, we just get $\vec{\beta}(t)$ to trace out a circle of radius a in the $xy-$plane. The curvature of this circle is $k(s) = \dfrac{a}{a^2 + b^2} = \dfrac{1}{a}$. Therefore, circles of "small" radius have "large" curvature and circles of "large" radius have "small" curvature (and the curvature of a straight line is 0). See Figure I-4.

Definition. For a given value of s, the circle of radius $\dfrac{1}{k(s)}$ which is tangent to $\vec{\alpha}$ and which lies in the plane of $\vec{T}(s)$ and $\vec{N}(s)$ is the *osculating circle* of $\vec{\alpha}$ at point $\vec{\alpha}(s)$. The center of the osculating circle is the *center of curvature* of $\vec{\alpha}$ at point $\vec{\alpha}(s)$, denoted $\vec{c}(s)$. The plane containing the osculating circle is the *osculating plane*. See Figure I-6.

Note. $\vec{c}(s)$ is calculated by going from point $\vec{\alpha}(s)$ a distance $\dfrac{1}{k(s)}$ in the direction $\vec{N}(s)$. That is,

$$\vec{c}(s) = \vec{\alpha}(s) + \frac{1}{k(s)}\vec{N}(s).$$

Example (p. 13, # 12). Consider the helix above parameterized in terms of s:

$$\vec{\alpha}(s) = \left(a \cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right), a \sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right), \frac{bs}{\sqrt{a^2 + b^2}}\right).$$

Find $\vec{c}(s)$ and show that it is also a helix.

Solution. The center of curvature is

$$\vec{c}(s) = \vec{\alpha}(s) + \frac{1}{k(s)}\vec{N}(s),$$

where, from Example 3, $k(s) = \dfrac{a}{a^2+b^2}$ and

$$\vec{N}(s) = \left(-\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0 \right).$$

So

$$\begin{aligned} \vec{c}(s) &= \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) - \frac{a^2+b^2}{a}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \right. \\ &\qquad \left. a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) - \frac{a^2+b^2}{a}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right) \\ &= \left(-\frac{b^2}{a}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\frac{b^2}{a}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right). \end{aligned}$$

If we let $A = \dfrac{-b^2}{a}$ and $t = \dfrac{s}{\sqrt{a^2+b^2}}$, then

$$\vec{c}(t) = (A\cos t, A\sin t, bt),$$

which is a circular helix.

Note. The curvature $k(s)$ of a curve $\vec{\alpha}(s)$ gives an idea of how a curve "twists" but does not provide a complete description of the curves "gyrations" (as the text puts it; see page 8). There is information in how the osculating plane tilts as s varies.

Recall. A plane in E^3 is determined by a point (x_0, y_0, z_0) and a normal vector $\vec{n} = (A, B, C)$ (we will not notationally distinguish between points and vectors). If (x, y, z) is a point in the plane, then a vector from (x_0, y_0, z_0) to (x, y, z) is perpendicular to $\vec{n}$ and so

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = (A, B, C) \cdot (x - x_0, y - y_0, z - z_0)$$

$$= A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

This can be rearranged as $Ax + By + Cz = D$ for some constant D. Notice that "twistings" of the plane would be reflected in changes in the direction of the normal vector.

Example. Find the equation of the plane through the points $(1, 2, 3)$, $(-2, 3, 3)$ and $(1, 2, 4)$.

Solution. The vectors $\vec{a} = (1 - (-2), 2 - 3, 3 - 3) = (3, 1, 0)$ and $\vec{b} = (1 - 1, 2 - 2, 4 - 3) = (0, 0, 1)$ both lie in the desired plane. Recall that in E^3, $\vec{a}$ and $\vec{b}$ are both orthogonal to $\vec{a} \times \vec{b}$ (provided $\vec{a}$ is not a scalar multiple of $\vec{b}$ - See Appendix A for more details). So we can take $\vec{n} = \vec{a} \times \vec{b}$ as a normal vector for the desired plane.

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i} - 3\vec{j} + 0\vec{k} = (1, -3, 0).$$

So the desired plane satisfies $1(x - 1) - 3(y - 2) + 0(z - 3) = 0$ or $x - 3y = -5$. Expressed parametrically,

$$\begin{aligned} x &= -5 + 3t \\ y &= t \\ z &= z \text{ (a "free variable").} \end{aligned}$$

(Again, we make no notational distinction between a vector and a point. There is a difference, though: points have locations, but vectors don't [nonzero vectors have a length and a direction, but no position].)

Definition. The *binormal vector* is $\vec{B} = \vec{T} \times \vec{N}$.

Note. $\vec{B}$ is orthogonal to both $\vec{T}$ and $\vec{N}$ (and therefore to the osculating plane). The derivative of $\vec{B}$ is

$$\vec{B}' = (\vec{T} \times \vec{N})' = \vec{T}' \times \vec{N} + \vec{T} \times \vec{N}'$$

(where $'$ represents derivative with respect to whatever the variable of parameterization is). Since $\vec{T}' = k\vec{N}$, $\vec{T}' \times \vec{N} = \vec{0}$ and so $\vec{B}' = \vec{T} \times \vec{N}'$. Since $\vec{N}$ is a unit vector, $\vec{N}'$ is perpendicular to $\vec{N}$ (as argued above for $\vec{T}$). Also, $\vec{T}$ is perpendicular to $\vec{N}$ since $\vec{N} = (1/k)\vec{T}'$. So both $\vec{T}$ and $\vec{N}'$ are perpendicular to $\vec{N}$ and so $\vec{B}' = \vec{T} \times \vec{N}'$ is a multiple of $\vec{N}$ (since we are in 3-dimensions), say $\vec{B}' = -\tau\vec{N}$. (Notice that if $\vec{T}$ and $\vec{N}'$ are multiples of each other, then $\tau = 0$.)

Definition. The *torsion* of $\vec{\alpha}$ at $\vec{\alpha}(s)$ is the function $\tau(s)$ where $\vec{B}'(s) = -\tau(s)\vec{N}(s)$.

Note. The torsion measures the twisting (or turning) of the osculating plane and therefore describes how much $\vec{\alpha}$ "departs from being a plane curve" (as the text says - see page 9).

Example. Calculate $\vec{B}$, $\vec{B}'$ and $\tau(s)$ for the helix above.

Solution. We have

$$\vec{T}(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\right)$$

and

$$\vec{N}(s) = -\left(\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right),$$

so

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) & a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) & b \\ -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) & -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), -b\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\right)$$

and

$$\vec{B}' = \frac{b}{a^2+b^2}\left(\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right).$$

Therefore, since $\vec{B}' = -\tau\vec{N}$, we have $\tau(s) = \dfrac{b}{a^2+b^2}$.

Note. Notice that the torsion is a constant in the previous example. This makes sense since the osculating plane tilts at a constant rate as a particle travels (uniformly) up the helix (or "spring"). With $b = 0$, the helix is, in fact, a circle in the $xy-$plane and so the osculating plane does not change and $\tau = 0$.

Lemma. If $\tau = 0$ at every point of a curve $\vec{\alpha}$, then $\vec{\alpha}$ lies in a plane.

Proof. Since $\vec{B}'(s) = -\tau(s)\vec{N}(s)$, then with $\tau(s) = 0$ we have $\vec{B}'(s) = \vec{0}$ and so $\vec{B}(s) = \vec{B}$ is a constant. Consider the plane containing $\vec{\alpha}(0)$ with normal vector $\vec{B}$. Let $\vec{\alpha}(s)$ be an arbitrary point on curve $\vec{\alpha}$ and define $f(s) = (\vec{\alpha}(s) - \vec{\alpha}(0)) \cdot \vec{B} = \vec{\alpha}(s)\cdot\vec{B} - \vec{\alpha}(0)\cdot\vec{B}$. Then $f'(s) = \vec{\alpha}'(s)\cdot\vec{B} = \vec{T}(s)\cdot\vec{B} = 0$ (since $\vec{B} \perp \vec{T}$). So $f(s)$ is a constant with $f(0) = (\vec{\alpha}(0) - \vec{\alpha}(0))\cdot\vec{B} = 0$. Therefore, $f(s) = 0$ for all s and so $f(s) = (\vec{\alpha}(s) - \vec{\alpha}(0))\cdot\vec{B} = 0$. So vector $\vec{\alpha}(s) - \vec{\alpha}(0)$ is orthogonal to $\vec{B}$ and "point" $\vec{\alpha}(s)$ lies in the plane containing point $\vec{\alpha}(0)$ and with normal vector $\vec{B}$. ∎

Note. We will see that the shape of a curve is completely determined by the curvature $k(s)$ and torsion $\tau(s)$.

Example (Exercise 14, page 14). Prove the Second Serret-Frenet Formula: $\vec{N}' = -k\vec{T} + \tau\vec{B}$. The First Serret-Frenet Formula is $\vec{T}' = k\vec{N}$ and the Third Serret-Frenet Formula is $\vec{B}' = -\tau\vec{N}$.

Proof. Since $\vec{N}$, $\vec{T}$, and $\vec{B}$ are mutually orthogonal, and each is a unit vector, we can write

$$\vec{\alpha} = (\vec{\alpha} \cdot \vec{N})\vec{N} + (\vec{\alpha} \cdot \vec{T})\vec{T} + (\vec{\alpha} \cdot \vec{B})\vec{B}.$$

Differentiating with respect to s:

$$\vec{\alpha}' = (\vec{\alpha}' \cdot \vec{N} + \vec{\alpha} \cdot \vec{N}')\vec{N} + (\vec{\alpha} \cdot \vec{N})\vec{N}'$$

$$+(\vec{\alpha}' \cdot \vec{T} + \vec{\alpha} \cdot \vec{T}')\vec{T} + (\vec{\alpha} \cdot \vec{T})\vec{T}' + (\vec{\alpha}' \cdot \vec{B} + \vec{\alpha} \cdot \vec{B}')\vec{B} + (\vec{\alpha} \cdot \vec{B})\vec{B}'.$$

So

$$\vec{\alpha}' = \vec{\alpha}' + (\vec{\alpha} \cdot \vec{N}')\vec{N} + (\vec{\alpha} \cdot k\vec{N})\vec{T} + (\vec{\alpha} \cdot (-\tau\vec{N}))\vec{B}$$

$$+(\vec{\alpha} \cdot \vec{N})\vec{N}' + (\vec{\alpha} \cdot \vec{T})k\vec{N} + (\vec{\alpha} \cdot \vec{B})(-\tau\vec{N}) \qquad (1)$$

using the First and Third Serret-Frenet Formulas. Now

$$\frac{d}{ds}[1] = \frac{d}{ds}\left[\|\vec{N}\|^2\right] = \frac{d}{ds}\left[\vec{N} \cdot \vec{N}\right] = 2\vec{N} \cdot \vec{N}' = 0.$$

So $\vec{N}$ and $\vec{N}'$ are orthogonal. Equating multiples of $\vec{N}$ in equation (1) (since $\vec{T}$ and $\vec{B}$ are also orthogonal to $\vec{N}$):

$$\vec{\alpha} \cdot (\vec{N}' + k\vec{T} - \tau\vec{B}) = 0. \qquad (2)$$

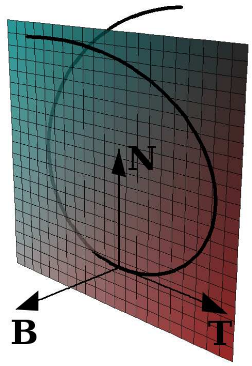
So equation (2) implies either $\vec{N}' = -k\vec{T} + \tau\vec{B}$, or $\vec{N}' + k\vec{T} - \tau\vec{B}$ is orthogonal to $\vec{\alpha}$. In the first case we are done, in the second case it must be that $\vec{\alpha} = a(s)\vec{N} = a\vec{N}$ (since $\vec{N}'$, $\vec{T}$ and $\vec{B}$ are all orthogonal to $\vec{N}$). Then $\vec{\alpha}' = a'\vec{N} + a\vec{N}' = \vec{T}$ implies that $\vec{T} = a\vec{N}'$ and $a'\vec{N} = \vec{0}$. So $a' = 0$ and $a(s) = a$ is constant. Therefore $\vec{\alpha}(s)$ lies on a sphere of radius $|a|$. Now $\vec{B} = \vec{T} \times \vec{N}$ so

$$\vec{B}' = \vec{T}' \times \vec{N} + \vec{T} \times \vec{N}' = (k\vec{N}) \times \vec{N} + (a\vec{N}') \times \vec{N}' = \vec{0}.$$

But $\vec{B}' = -\tau\vec{N}$ so $\tau = 0$. Therefore, as commented in Exercise 1.1.13, $\vec{\alpha}(s)$ is planar. So $\vec{\alpha}(s)$ is a circle of radius $|a|$. In this case, $\vec{\alpha} = a\vec{N}$ and $k = 1/a$. Since $\tau = 0$,

$$k\vec{T} - \tau\vec{B} = k\vec{T} = \frac{1}{a}\vec{T} = \frac{1}{a}(a\vec{N}') = \vec{N}'.$$

In either case, the result holds. (Note: The second case occurs in the case of circular motion: consider $\vec{\alpha}(t) + (a\cos t, a\sin t, 0)$.) ∎



Serret Frenet Frame
(from Wikipedia)

Joseph Alfred Serret
(from MacTutor)

Note. The Serret-Frenet formulas were independently discovered by two French mathematicians, Jean Frédéric Frenet (1816–1900) and Joseph Alfred Serret (1819-1885). Frenet included these ideas in his doctoral thesis of 1847. Notice that the three Serret-Frenet formulas are vector formulas in 3-dimensions, and so actually correspond to nine (component) formulas. Frenet stated only six of the scalar formulas. He published his results in a paper titled *Sur quelque propriétés des courbes à double courbure* in the *Journal de mathematique pures et appliques* in 1852. See the MacTutor History of Mathematics biography of Frenet at `http://www-history.mcs.st-andrews.ac.uk/Biographies/Frenet.html`.

Note. Serret stated all nine of the (component) formulas in 1851. He also published papers on number theory, calculus, the theory of functions, group theory, mechanics, differential equations, and astronomy. See the MacTutor History of Mathematics biography of Serret at `http://www-history.mcs.st-andrews.ac.uk/Biographies/Serret.html`.

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