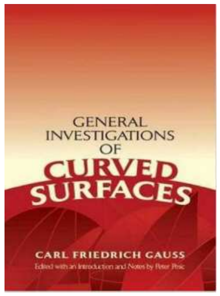
1.2 Gauss Curvature (Informal Treatment)



Curved Surfaces

Carl Friederich Gauss

Note. The material of Sections 1.2 through 1.8 is due to Carl Friederich Gauss (1777-1855). Gauss's original work is currently in print by Dover Publications (Mineola, NY 2005) in *General Investigations of Curved Surfaces.* This brief book (130 pages) includes the 1827 "General Investigations of Curved surfaces" (*Disquisitiones generales circa superficies curvas*) and the 1825 "New General Investigations of Curved Surfaces." There are several pages of notes and additional notes. The Dover book is a reprint of the Princeton University Library version of 1902 which was translated from the original Latin and German by Adam Hiltebeitel and James Morehead. A historical introduction (on which the historical material in these notes is based) is given by Peter Pesic. The images for this section are from the MacTutor History of Mathematics archive.

Georg Friederich Bernhard Riemann

Note. Gauss's work on surfaces was generalized by his student, Georg Friederich Bernhard Riemann (1826-1866), to n-dimensional manifolds. This laid the foundations needed for Albert Einstein (1879–1955) to use curvature of spacetime in his work on general relativity. According to Einstein, "the importance of Gauss for the development of modern physical theory and especially for the mathematical fundamentals of the theory of relativity is overwhelming indeed; ... If he had not created his geometry of surfaces, which served Riemann as a basis, it is scarcely conceivable that anyone else would have discovered it." The Einstein quote is credited to the source: Tord Hall, *Carl Friederich Gauss*, translated by Albert Froderberg. Cambridge, Mass.: MIT Press, 1970 (see page iii or the Dover book).

Note. In this section, we actually follow the approach to curvature given by Leonhard Euler (1707-1783). In 1760, Euler uses the product of a maximum and minimum radius of curvature at a point (signed positively or negatively according to convexity or concavity) to define the curvature as the product of the reciprocals of these extrema.

Note. Gauss worked from 1818 to 1832 directing the surveying of the Electorate of Hanover. Though this took him away from his pure math research, it involved triangulations and extensive calculations. This experience with mapping the surface of the Earth no doubt impacted Gauss's study of surfaces. In his mathematical work, Gauss begins with the perspective of a surveyor concerned with the directions of various straight lines in space he specifies through the use of an 'auxiliary sphere' of unit radius. In 1771, Euler introduced coordinates on a curved surface in his paper *De solidus quorum superficiem in planum explicare licet.* In fact, Gauss uses these coordinates (now called "Gaussian coordinates") in his exploration of curvature of surfaces and uses them to define the *measure of curvature* (now called the "Gaussian Curvature") as the ratio between an infinitesimal area on the curved surface and the corresponding area mapped to the auxiliary sphere. We will see this result in Section 1.6 ("Gauss Curvature in Detail") when we show that the curvature at a point P on a surface is

$$K(P) = \lim_{\Omega \to (u_0^1, u_0^2)} \frac{\text{Area } \vec{U}(\Omega)}{\text{Area } \vec{X}(\Omega)}.$$

Note. Gauss's *Theorema Egregium* (which we see in Section 1.8) shows that a surface can have an *intrinsic* curvature that does not change even if that surface is *extrinsically* bent (always leaving distances invariant) such as a cylinder and plane, both of which have curvature 0. Gauss's theorem has a surprising implication, that the curvature of a surface is an *intrinsic* property, as opposed to a property of how the surface is imbedded in an ambient space. See pages v and vi of the Dover book.

Note. Gauss introduced the use of a *metric*, an expression for the infinitesimal distance between points that generalizes the Pythagorean formula to curved surfaces expressed in arbitrary coordinates. This was the starting point Riemann took in 1854 to generalize Gauss's work to a "space" of n dimensions and determine its intrinsic curvature. Riemann's work appears in "On the Hypotheses that Lie at the Foundations of Geometry," see William Ewald, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*. Oxford: Clarendon Press (1996), pages 649–661. Gauss also developed Euler's work on *geodesics*, the shortest possible lines on a curved surface, which are the natural generalizations of straight lines in the Euclidean plane. See page vi of the Dover book.

Johann Bolyai

Carl Gauss

Nicolai Lobachevsky

Note. (From *Non-Euclidean Geometry* by Roberto Bonola, Dover Publications, 1955.) Historically, it is recognized that there are three founders of hyperbolic geometry: Carl Frederick Gauss (1777–1855), Nicolai Lobachevsky (1793–1856), and Johann Bolyai (1802–1860). Historical documents (primarily in the form of letters from Gauss to other mathematicians) seem to indicate that Gauss was the first to understand that there was an alterative to Euclidean geometry. Quoting Bonola: "Gauss was the first to have a clear view of geometry independent of the Fifth Postulate, but this remained quite fifty years concealed in the mind of the great geometer, and was only revealed after the works of Lobatschewsky (1829–30) and J. Bolyai (1832) appeared." Based on various passages in Gauss's letters, one can tell that he began his thoughts on non-Euclidean geometry in 1792. However, Gauss did not recognize the existence of a logically sound non-Euclidean geometry by intuition or a flash of genius, but only after years of thought in which he overcame the then universal prejudice against alternatives to the Parallel Postulate. Letters show that Gauss developed the fundamental theorems of the new geometry some time shortly after 1813. However, in his correspondence, Gauss pleaded with his colleagues to keep silent on his results. However, "throughout his life he failed to publish anything on the matter" [*An Introduction to the History of Mathematics*, 5th Edition, by Howard Eves, Saunders College Publishing, 1983]. The honor of being the first to publish work on non-Euclidean geometry goes to others. For more information on the history and content of non-Euclidean geometry (the case of hyperbolic geometry), see the online presentation at `http://faculty.etsu.edu/gardnerr/noneuclidean/hyperbolic.pdf`. There can be no doubt that Gauss's experience with surveying and curved surfaces influenced (or was influenced by) his thoughts on non-Euclidean geometry.

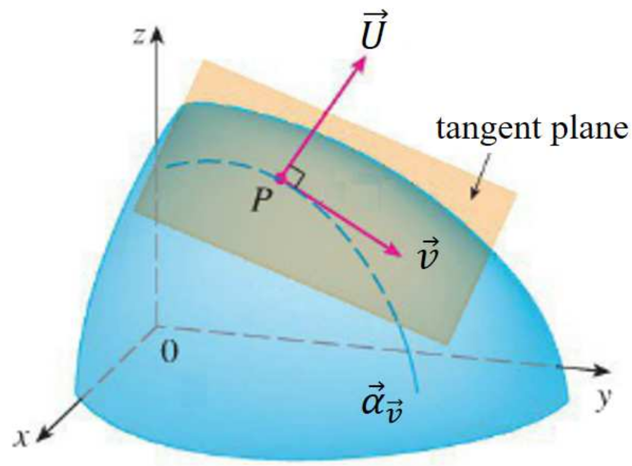
Note. We now return to Faber's text. The approach in this section uses Euler's idea of of the curvature of a surface as a product of certain cross sectional curvatures (though we will still call this "Gauss curvature"). It is shown in Section 1.6 that this is equivalent to Gauss's approach based on a limit of a ratio of areas.

Recall. If $f(x, y, z)$ is a (scalar valued) function, then for c a constant, $f(x, y, z) = c$ determines a *surface* (we assume all second partials of f are continuous and so the surface is *smooth*). The *gradient* of f is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

If $\vec{v}_0$ is a vector tangent to the surface $f(x, y, z) = c$ at point $\vec{P}_0 = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0)$ is orthogonal to $\vec{v}_0$ (and so ∇f is orthogonal to the surface). The equation of a plane tangent to the surface can be calculated using ∇f as the normal vector for the plane.

Definition. Let $\vec{v}$ be a unit vector tangent to a smooth surface $M \subset E^3$ at a point $\vec{P}$ (again making no distinction between a vector and a point). Let $\vec{U}$ be a unit vector normal (perpendicular) to M at point $\vec{P}$. The plane through point $\vec{P}$ which contains vectors $\vec{v}$ and $\vec{U}$ intersects the surface in a curve $\vec{\alpha}_{\vec{v}}$ called the *normal section of M at $\vec{P}$ in the direction $\vec{v}$.*



Based on an image from `https://math.boisestate.edu/~jaimos/classes/m275-spring2017/notes/gradient2.html`

Example. Find the normal section of $M : x^2 + y^2 = 1$ (an infinitely tall right circular cylinder of radius 1) at the point $\vec{P} = (1, 0, 0)$ in the direction $\vec{v} = (0, 1, 0)$.

Solution. A normal vector to M at $\vec{P}$ is

$$\nabla(x^2 + y^2) = (2x, 2y, 0)|_{(1,0,0)} = (2, 0, 0).$$

Therefore, we take $\vec{U} = (1, 0, 0)$. The plane containing $\vec{U}$ and $\vec{v}$ has as a normal vector

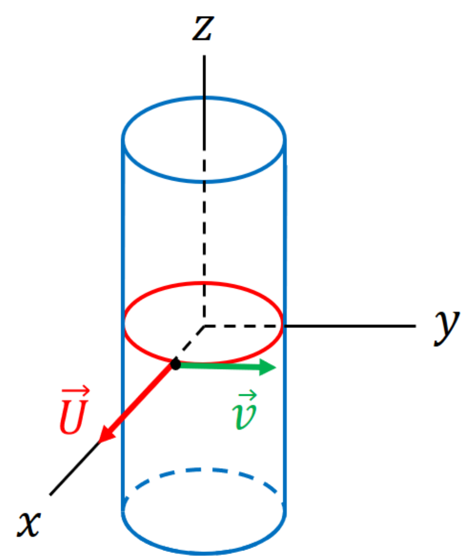
$$\vec{U} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1).$$

Therefore the equation of this plane is

$$0(x-1) + 0(y-0) + 1(z-0) = 0$$

or $z = 0$. The intersection of this plane and the surface is

$$\vec{\alpha}_{\vec{v}} = \{(x, y, z) \mid x^2 + y^2 = 1, z = 0\}.$$



Note. Each normal section $\vec{\alpha}_{\vec{v}}$ to a surface can be approximated by a circle (as in the previous section). Recall that if a plane curve has a curvature k at some point $\vec{P}$, then this osculating circle has radius $1/k$ and its center is located $1/k$ units from $\vec{P}$ in the direction of the principal normal vector $\vec{N}$.

Definition. Let $\vec{\alpha}_{\vec{v}}$ be a normal section to a smooth surface M at point $\vec{P}$ in the direction $\vec{v}$. Let $\vec{U}$ be a unit normal to M at $\vec{P}$ ($-\vec{U}$ is also a unit normal to M at $\vec{P}$). The *normal curvature of M at $\vec{P}$ in the $\vec{v}$ direction with respect to $\vec{U}$*, denoted $k_{n,\vec{U}}(\vec{v})$, is

$$k_{n,\vec{U}}(\vec{v}) = \frac{\vec{U} \cdot \vec{N}}{R(\vec{v})}$$

where $\vec{N}$ is the principal normal vector of $\vec{\alpha}_{\vec{v}}$ at $\vec{P}$ and $R(\vec{v})$ is the radius of the osculating circle to $\vec{\alpha}_{\vec{v}}$ at $\vec{P}$. If $\vec{\alpha}_{\vec{v}}$ has zero curvature at $\vec{P}$, we take $k_{n,\vec{U}}(\vec{v}) = 0$.

Note. If $\vec{U}$ and $\vec{N}$ are parallel, then

$$k_{n,\vec{U}}(\vec{v}) = \frac{1}{R(\vec{v})}$$

and if $\vec{U}$ and $\vec{N}$ are antiparallel (i.e. point in opposite directions) then

$$k_{n,\vec{U}}(\vec{v}) = \frac{-1}{R(\vec{v})}.$$

So, $\left|k_{n,\vec{U}}(\vec{v})\right|$ is just the curvature of $\vec{\alpha}_{\vec{v}}$ at $\vec{P}$. The text does not include the vector $\vec{U}$ in its notation, but our approach is equivalent to its.

Example. What is $k_{n,\vec{U}}(\vec{v})$ for the cylinder $x^2 + y^2 = 1$ at $\vec{P} = (1, 0, 0)$ in the direction $\vec{v} = (0, 1, 0)$ with respect to $\vec{U} = (1, 0, 0)$?

Solution. As we saw in the previous example,

$$\vec{\alpha}_{\vec{v}} = \{(x, y, z) \mid x^2 + y^2 = 1, z = 0\}.$$

We can parameterize $\vec{\alpha}_{\vec{v}}$ as

$$\vec{\alpha}(s) = (\cos s, \sin s, 0)$$

where $s \in [0, 2\pi]$. Then

$$\vec{T}(s) = \vec{\alpha}'(s) = (-\sin s, \cos s, 0)$$

and

$$\vec{T}'(s) = \vec{\alpha}''(s) = (-\cos s, -\sin s, 0).$$

At point $\vec{P}$, $s = 0$, so the principal normal vector at $\vec{P}$ is

$$\vec{N}(0) = \vec{T}'(0)/\|\vec{T}'(0)\| = (-1, 0, 0).$$

Therefore

$$k_{n,\vec{U}}(\vec{v}) = \frac{\vec{U} \cdot \vec{N}}{R(\vec{v})} = \frac{-1}{1} = -1.$$

Note. We will see in Section 1.6 that the normal curvature of M at $\vec{P}$ in the $\vec{v}$ direction with respect to $\vec{U}$ assumes a maximum and a minimum in directions $\vec{v}_1$ and $\vec{v}_2$ (respectively) which are orthogonal.

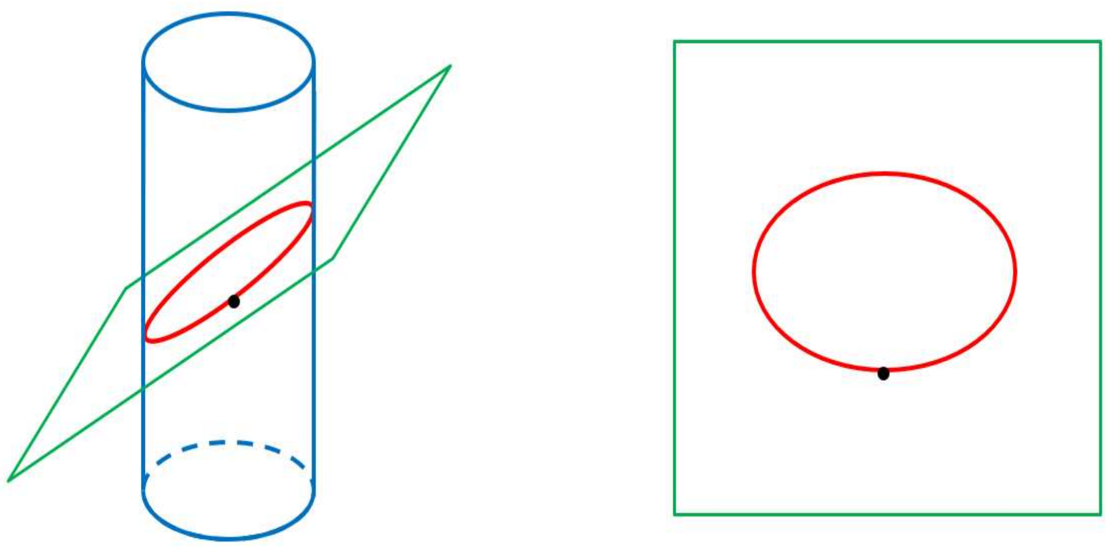
Definition. The directions $\vec{v}_1$ and $\vec{v}_2$ described above are the *principal directions* of M at $\vec{P}$. Let k_1 and k_2 be the maximum and minimum values (respectively) of $k_{n,\vec{U}}(\vec{v})$ at $\vec{P}$ (we can take $\vec{v} = \vec{v}_1$ and $\vec{v} = \vec{v}_2$, respectively). Then k_1 and k_2 are the *principal curvatures* of M at $\vec{P}$. The product $k_1 k_2$ is the *Gauss curvature* of M at $\vec{P}$, denoted $K(\vec{P})$:

$$K(\vec{P}) = k_1 k_2.$$

Note. Even though $k_{n,\vec{U}}(\vec{v})$ depends on the choice of $\vec{U}$, $K(\vec{P})$ is independent of the choice of $\vec{U}$ (if we use $-\vec{U}$ for the normal to the surface instead of $\vec{U}$, we change the sign of $k_{n,\vec{U}}(\vec{v})$ and so the product $k_1 k_2$ remains the same).

Example 4 (page 15). Evaluate $K(\vec{P})$ for the right circular cylinder $x^2 + y^2 = 1$ at $\vec{P} = (1, 0, 0)$.

Solution. At $\vec{P}$, $\vec{\alpha}_{\vec{v}}$ is an ellipse with semi-minor axis 1, unless $\vec{v} = (0, 0, \pm 1)$:



With $\vec{U} = (1, 0, 0)$ (and $\vec{N} = (-1, 0, 0)$ which implies nonpositive $k_{n,\vec{U}}(\vec{v})$) we have a minimum value of normal curvature of $k_2 = -1$ (as given in the previous example - this value is attained when $\vec{\alpha}_{\vec{v}}$ is a circle). Now with $\vec{v} = (0, 0, \pm 1)$ we get that $\vec{\alpha}_{\vec{v}}$ is a pair of parallel lines and then $k_{n,\vec{U}}(\vec{v}) = 0$ (recall the curvature of a line is 0). So

$$K(\vec{P}) = (-1)(0) = 0.$$

Note. The Gauss curvature of a cylinder is 0 at every point. This is also the case for a plane. An INFORMAL reason for this is that a cylinder can be cut and peeled open to produce a plane (and conversely) without stretching or tearing (other than the initial cut) and without affecting lengths (such an operation is called an *isometry*).

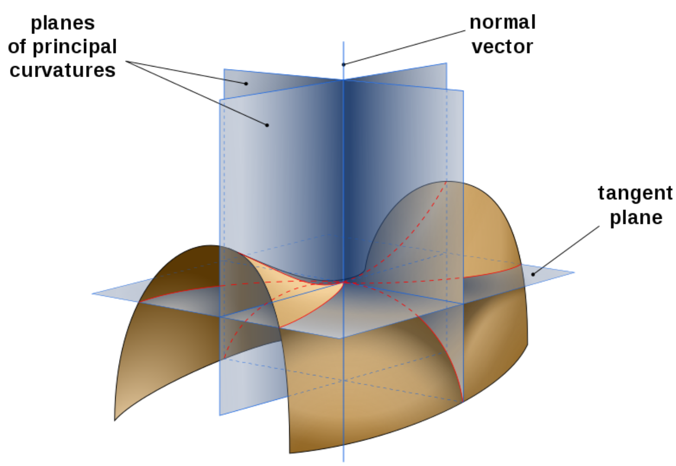
Example. A sphere of radius r has normal curvature at every point of $k_{n,\vec{U}}(\vec{v}) = \pm 1/r$ (depending on the choice of $\vec{U}$) and so the Gauss curvature is $K = 1/r^2$.

Note. A surface M has positive curvature at point $\vec{P}$ if, in a deleted neighborhood of $\vec{P}$ on M, all points lie on the same side of the plane tangent to M at $\vec{P}$. If for all neighborhoods of $\vec{P}$ on M, some points are on one side of the tangent plane and some points are on the other side, then the surface has negative curvature (this will be made more rigorous latter).

Example 5 (page 18). The hyperbolic paraboloid

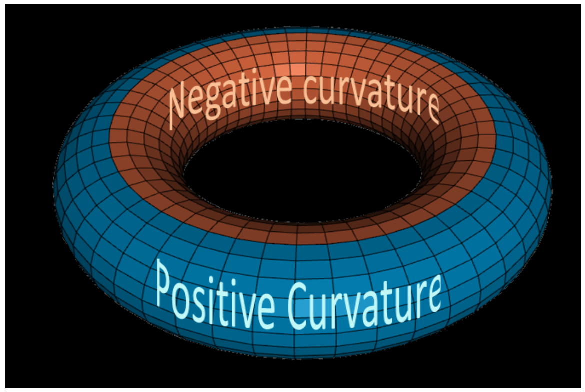
$$z = \frac{1}{2}(y^2 - x^2)$$

has negative curvature at each point.



From the Wikipedia page on “Gaussian Curvature.”

Example 6 (page 19). A torus has some points with positive curvature, some with negative curvature and some with 0 curvature.



From the Wikipedia page on "Gaussian Curvature."

Note. We have defined curvature as an *extrinsic* property of a surface (using things external to the surface such as normal vectors). We will see in Gauss's *Theorema Egregium* that we can redefine curvature as an *intrinsic* property which can be measured only using properties of the surface itself and not using any properties of the space in which the surface is embedded. This will be important when we address the questions as to whether the universe is open or closed (and whether it has positive, zero, or negative curvature).

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