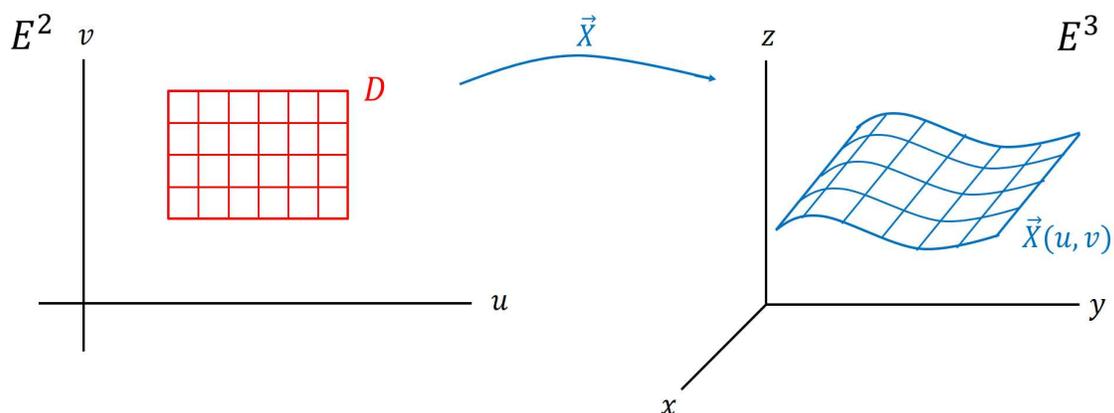


1.3 Surfaces in E^3

Note. A surface M may be described as the image of a subset D of \mathbb{R}^2 under a vector valued function of two variables

$$\vec{X}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

When using this notation, we assume x, y, z have continuous partial derivatives up to the third order.



Definition. A surface given as above is *regular* if the vectors

$$\vec{X}_1(u, v) = \frac{\partial \vec{X}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$\vec{X}_2(u, v) = \frac{\partial \vec{X}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

are linearly independent for each $(u, v) \in D$.

Note. \vec{X}_1 and \vec{X}_2 are linearly independent on D is equivalent to the property:

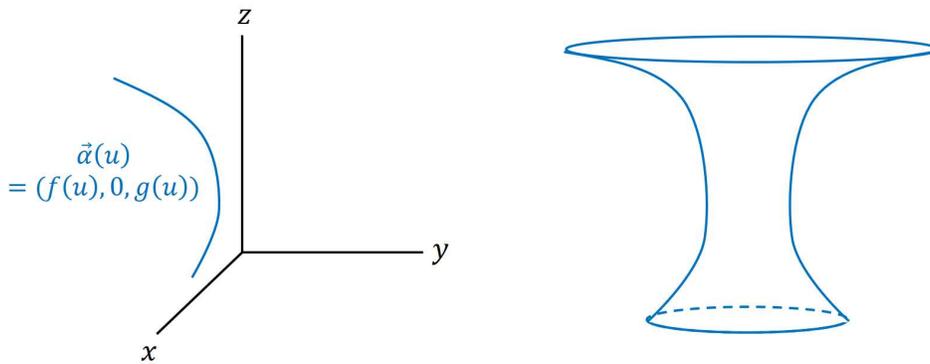
$$\vec{X}_1 \times \vec{X}_2 \neq \vec{0} \text{ for all } (u, v) \in D.$$

Note. The condition of regularity insures that \vec{X} is one-to-one and has a continuous inverse.

Example (Exercise 1.3.1(a)). If a smooth curve of the form $\vec{\alpha}(u) = (f(u), 0, g(u))$ in the xz -plane is revolved about the z -axis, the resulting *surface of revolution* is given by

$$\vec{X}(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$

Show that \vec{X} is regular provided $f(u) \neq 0$ and $\vec{\alpha}'(u) \neq \vec{0}$ for all u .



Solution. Well,

$$\vec{X}_1(u, v) = \frac{\partial \vec{X}}{\partial u} = (f'(u) \cos v, f'(u) \sin v, g'(u))$$

$$\vec{X}_2(u, v) = \frac{\partial \vec{X}}{\partial v} = (-f(u) \sin v, f(u) \cos v, 0).$$

If $\vec{\alpha}'(u) \neq 0$ for all u , then for a given u , either $g'(u) \neq 0$ or $f'(u) \neq 0$. If $g'(u) \neq 0$ then \vec{X}_1 and \vec{X}_2 are linearly independent (in the third component). If $f'(u) \neq 0$ and $g'(u) = 0$, consider:

$$\vec{X}_1 \times \vec{X}_2 = f'(u)f(u)(\cos^2 v + \sin^2 v)\vec{k} = f'(u)f(u)\vec{k}.$$

Since $f(u) \neq 0$ for all u , $f'(u)f(u) \neq 0$ and so $\vec{X}_1 \times \vec{X}_2 \neq \vec{0}$ and \vec{X}_1 and \vec{X}_2 are linearly independent.

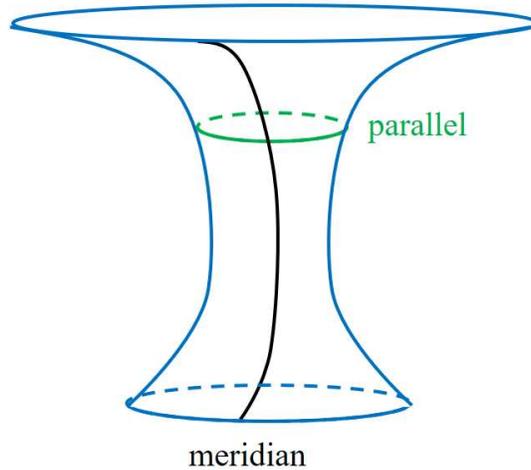
Definition. A vector \vec{v} is a *tangent vector* to surface M at point \vec{P} if there is a curve on M which passes through \vec{P} and has velocity vector \vec{v} at \vec{P} . The set of all tangent vectors to M at \vec{P} is the *tangent plane* of M at \vec{P} , denoted $T_{\vec{P}}M$.

Note. $T_{\vec{P}}M$ is a 2-dimensional vector space with $\{\vec{X}_1(u_0, v_0), \vec{X}_2(u_0, v_0)\}$ as a basis, where $\vec{X}(u_0, v_0) = \vec{P}$.

Definition. The curve $\vec{X}(u, v_0)$ is a *u-parameter curve* and $\vec{X}(u_0, v)$ is a *v-parameter curve* of surface M (u_0 and v_0 are constants).

Note. $\vec{X}_1(u_0, v_0)$ is a velocity vector of $\vec{X}(u, v_0)$ and $\vec{X}_2(u_0, v_0)$ is a velocity vector of $\vec{X}(u_0, v)$.

Example (Exercise 1.3.1(b)). Consider the surface of revolution of Exercise 1.3.1(a). Describe the *u*- and *v*-parameter curves and show they intersect orthogonally (the *u*-parameter curves are called *meridians* and the *v*-parameter curves are called *parallels*).



Solution. A u -parameter curve is of the form $(f(u) \cos v_0, f(u) \sin v_0, g(u))$ and has direction $\vec{X}_1(u, v_0) = (f'(u) \cos v_0, f'(u) \sin v_0, g'(u))$. A v -parameter curve is of the form $(f(u_0) \cos v, f(u_0) \sin v, g(u_0))$ and has direction $\vec{X}_2(u_0, v) = (-f(u_0) \sin v, f(u_0) \cos v, 0)$. If a u -parameter curve and a v -parameter curve intersect at (u_0, v_0) then at this point of intersection

$$\begin{aligned} \vec{X}_1(u_0, v_0) \cdot \vec{X}_2(u_0, v_0) &= -f(u_0)f'(u_0) \cos v_0 \sin v_0 \\ &\quad + f(u_0)f'(u_0) \cos v_0 \sin v_0 + g'(u_0) (0) = 0. \end{aligned}$$

Therefore the u -parameter and v -parameter curves are orthogonal when they intersect.

Example (Exercise 1.3.2(e)). For the surface of revolution

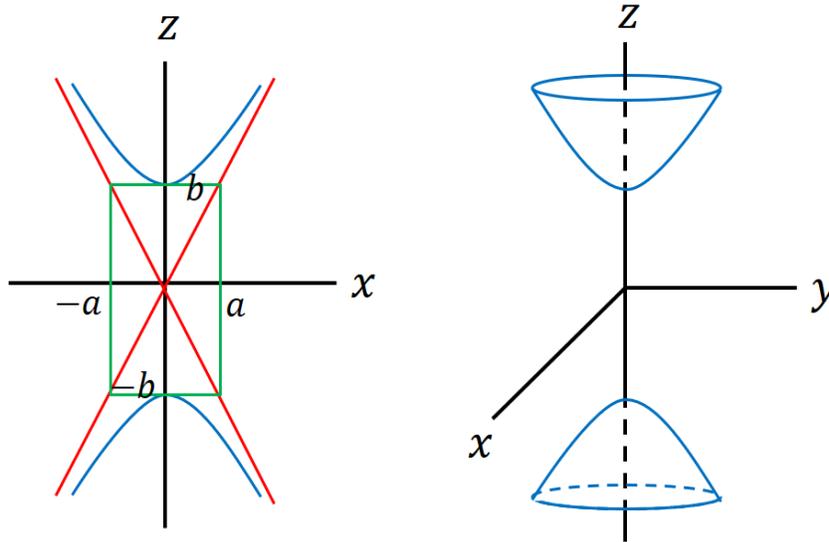
$$\vec{X}(u, v) = (a \sinh u \cos v, a \sinh u \sin v, b \cosh u), \quad u \neq 0,$$

sketch the profile curve ($v = 0$) in the xz -plane, and then sketch the surface. Prove that \vec{X} is regular and give an equation for the surface of the form $g(x, y, z) = 0$.

Solution. For the profile, with $v = 0$ we have $x = a \sinh u$ and $z = b \cosh u$. Since $\cosh^2 u - \sinh^2 u = 1$, we have

$$\left(\frac{z}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1, \quad x \neq 0.$$

So the profile and surface are:



Next,

$$\begin{aligned} \vec{X}_1 &= \frac{\partial \vec{X}}{\partial u} = (a \cosh u \cos v, a \cosh u \sin v, b \sinh u) \\ \vec{X}_2 &= \frac{\partial \vec{X}}{\partial v} = (-a \sinh u \sin v, a \sinh u \cos v, 0). \end{aligned}$$

So \vec{X}_1 and \vec{X}_2 are linearly independent since $b \sinh u \neq 0$ for $u \neq 0$. Therefore \vec{X} is regular. Since

$$\begin{aligned} x &= a \sinh u \cos v \\ y &= a \sinh u \sin v \\ z &= b \cosh u \end{aligned}$$

then $g(x, y, z) = a^2 z^2 - (b^2 x^2 + b^2 y^2) - a^2 b^2 = 0$ is the equation of the surface.

Revised: 6/8/2019