1.4 The First Fundamental Form

Note. Suppose M is a surface determined by $\vec{X}(u,v) \subset E^3$ and suppose $\vec{\alpha}(t)$ is a curve on M, $t \in [a,b]$. Then we can write $\vec{\alpha}(t) = \vec{X}(u(t),v(t))$ (then $(u(t),v(t))$ is a curve in $\mathbb{R}^2$ whose image under $\vec{X}$ is $\vec{\alpha}$). Then

$$\vec{\alpha}'(t) = \frac{\partial \vec{X}}{\partial u}\frac{du}{dt} + \frac{\partial \vec{X}}{\partial v}\frac{dv}{dt} = u'\vec{X}_1 + v'\vec{X}_2.$$

If $s(t)$ represents the arc length along $\vec{\alpha}$ (with $s(a) = 0$) then

$$s(t) = \int_a^t \|\vec{\alpha}'(r)\| dr$$

and

$$\frac{ds}{dt} = \|\vec{\alpha}'(t)\|$$

so

$$\left(\frac{ds}{dt}\right)^2 = \|\vec{\alpha}'(t)\|^2 = \vec{\alpha}' \cdot \vec{\alpha}' = (u'\vec{X}_1 + v'\vec{X}_2) \cdot (u'\vec{X}_1 + v'\vec{X}_2)$$
$$= u'^2(\vec{X}_1 \cdot \vec{X}_1) + 2u'v'(\vec{X}_1 \cdot \vec{X}_2) + v'^2(\vec{X}_2 \cdot \vec{X}_2).$$

Following Gauss' notation (briefly) we denote

$$E = \vec{X}_1 \cdot \vec{X}_1, \qquad F = \vec{X}_1 \cdot \vec{X}_2, \qquad G = \vec{X}_2 \cdot \vec{X}_2$$

and have

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

or in differential notation

$$ds^2 = E(du)^2 + 2F(du\,dv) + G(dv)^2.$$

Definition. Let M be a surface determined by $\vec{X}(u,v)$. The *first fundamental form* (or more commonly *metric form*) of M is $\left(\frac{ds}{dt}\right)^2$ or $(ds)^2$ as defined above.

Definition. A property of a surface which depends only on the metric form of the surface is an *intrinsic property*.

Note. The idea of an intrinsic property is that a "resident" of the surface can detect such a property without appealing to a "larger space" in which the surface is embedded. Certainly an inhabitant of a surface can measure distance within the surface.

Example 10, page 32. Consider the $xy-$plane described as $\vec{X}(u,v) = (u,v,0)$ where $u \in \mathbb{R}$ and $v \in \mathbb{R}$. Then $\vec{X}_1 = (1,0,0)$ and $\vec{X}_2 = (0,1,0)$. So

$$E = \vec{X}_1 \cdot \vec{X}_1 = 1, \qquad F = \vec{X}_1 \cdot \vec{X}_2 = 0, \qquad G = \vec{X}_2 \cdot \vec{X}_2 = 1.$$

Then the first fundamental form is

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2$$

or, in terms of x and y:

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Of course, this is the "usual" expression for the differential of arclength from Calculus 2.

Definition. The *matrix of the first fundamental form* of a surface M determined by $\vec{X}(u,v)$ is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where E, F, G are as defined as above.

Note. Since $\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$ and $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$, then

$$\begin{aligned}\|\vec{X}_1\times\vec{X}_2\|^2 &= \|\vec{X}_1\|^2\|\vec{X}_2\|^2\sin^2\theta = \|\vec{X}_1\|^2\|\vec{X}_2\|^2(1-\cos^2\theta)\\ &= \|\vec{X}_1\|^2\|\vec{X}_2\|^2 - (\|\vec{X}_1\|\|\vec{X}_2\|\cos\theta)^2\\ &= (\vec{X}_1\cdot\vec{X}_1)\cdot(\vec{X}_2\cdot\vec{X}_2) - (\vec{X}_1\cdot\vec{X}_2)^2\\ &= EG - F^2 = \begin{vmatrix} E & F \\ F & G\end{vmatrix} = g.\end{aligned}$$

Hence, $\|\vec{X}_1\times\vec{X}_2\| = \sqrt{g}$.

Note. This matrix determines dot products of tangent vectors. If $\vec{v} = a\vec{X}_1 + b\vec{X}_2$ and $\vec{w} = c\vec{X}_1 + d\vec{X}_2$ are vectors tangent to a surface M at a given point, then

$$\vec{v}\cdot\vec{w} = (a\vec{X}_1 + b\vec{X}_2)\cdot(c\vec{X}_1 + d\vec{X}_2) = Eac + F(ad+bc) + Gbd$$

$$= (a,b)\begin{pmatrix} E & F \\ F & G\end{pmatrix}\begin{pmatrix} c \\ d\end{pmatrix}.$$

Notation. We now replace the parameters u and v with u^1 and u^2. We then have

$$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij}du^idu^j$$

where the summation is taken (throughout this chapter) over the set $\{1,2\}$. In Chapter 3, we will sum over $\{1,2,3,4\}$. If $\vec{v}$ is a vector tangent to M at a point $\vec{P}$ and $\vec{v} = (v^1, v^2)$ in the basis $\{\vec{X}_1, \vec{X}_2\}$ for the tangent plane at $\vec{P}$, then we have

$$\vec{v} = \sum_i v^i\vec{X}_i.$$

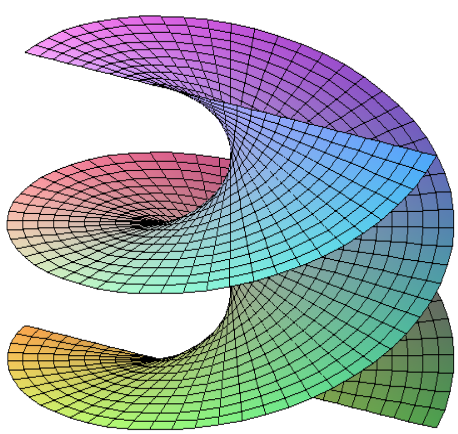
If $\vec{\alpha}(t)$ is a curve on M where $\vec{\alpha}$ is represented by $\vec{X}(u^1(t), u^2(t))$ then

$$\vec{\alpha}'(t) = u^{1\prime}(t)\vec{X}_1 + u^{2\prime}(t)\vec{X}_2 = \sum_i u^{i\prime}\vec{X}_i.$$

Notation. We denote the ij entry of $(g_{ij})^{-1}$ as g^{ij}. Therefore $(g_{ij})(g^{ij}) = \mathcal{I}$ and $\sum_j g_{ij}g^{jk} = \delta_i^k$ (the ik entry of $\mathcal{I}$) where

$$\delta_i^k = \begin{cases} 1 & \text{if } \quad i = k \\ 0 & \text{if } \quad i \neq k. \end{cases}$$

Example (Exercise 1.4.3(c)). For the surface $\vec{X}(u,v) = (u\cos v, u\sin v, bv)$ (the helicoid of Example 9), compute the matrix (g_{ij}), its determinate g, the inverse matrix (g^{ij}) and the unit normal vector $\vec{U}$.



From Wikipedia's page

`https://commons.wikimedia.org/wiki/File:Helicoid_JD.png`

Solution. Well

$$\begin{aligned}\vec{X}_1 &= \frac{\partial \vec{X}}{\partial u} = (\cos v, \sin v, 0)\\ \vec{X}_2 &= \frac{\partial \vec{X}}{\partial v} = (-u\sin v, u\cos v, b)\end{aligned}$$

and so

$$\begin{aligned}g_{11} &= \vec{X}_1 \cdot \vec{X}_1 = \cos^2 v + \sin^2 v + 0 = 1\\ g_{22} &= \vec{X}_2 \cdot \vec{X}_2 = u^2\sin^2 v + u^2\cos^2 v + b^2 = u^2 + b^2\\ g_{12} &= \vec{X}_1 \cdot \vec{X}_2 = -u\cos v\sin v + u\cos v\sin v + 0 = 0 = g_{21}.\end{aligned}$$

Therefore

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^2+b^2\end{pmatrix}$$

and $g = \det(g_{ij}) = u^2 + b^2$. Then

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{pmatrix}$$

$$= \frac{1}{u^2+b^2}\begin{pmatrix} u^2+b^2 & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2+b^2}\end{pmatrix}.$$

Now the unit normal vector is $\vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}$ and

$$\vec{X}_1 \times \vec{X}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & b\end{vmatrix}$$

$$= (b\sin v, -b\cos v, u\cos^2 v + u\sin^2 v) = (b\sin v, -b\cos v, u).$$

Now

$$\|\vec{X}_1 \times \vec{X}_2\| = \sqrt{b^2\sin^2 v + b^2\cos^2 v + u^2} = \sqrt{b^2+u^2}.$$

Therefore

$$\vec{U} = \left(\frac{b \sin v}{\sqrt{b^2+u^2}}, \frac{-b \cos v}{\sqrt{b^2+u^2}}, \frac{u}{\sqrt{b^2+u^2}} \right).$$

Definition. Suppose Ω is a closed subset of the u^1u^2-plane and that $\vec{X} : \Omega \to E^3$ is smooth (i.e. has continuous first partials), is one-to-one and regular (i.e. $\vec{X}_1$ and $\vec{X}_2$ are linearly independent) on the interior of Ω. Then the *area* of the surface $\vec{X}(\Omega)$ is

$$A = \int\int_{\Omega} \|\vec{X}_1 \times \vec{X}_2\| du^1 du^2 = \int\int_{\Omega} \sqrt{g} du^1 du^2.$$

(See page 37 of the text for motivation of this definition.)

Example (Exercise 1.4.6). (a) Show that the area A of the surface of revolution $\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ where $u \in [a,b]$ and $v \in [0, 2\pi]$ is given by

$$A = 2\pi \int_a^b |f(u)| \sqrt{f'(u)^2 + g'(u)^2}\, du.$$

(b) Show that the area of the surface obtained by revolving the graph $y = f(x)$ for $x \in [a,b]$ about the $x-$axis is given by

$$A = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\, dx.$$

Solution. (a) Consider the surface area of the surface of revolution $\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$. We have (from Exercise 1.4.5)

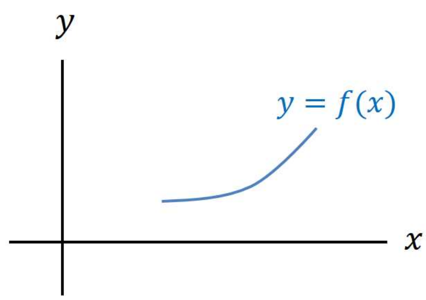
$$\|\vec{X}_1 \times \vec{X}_2\| = |f(u)| \sqrt{f'(u)^2 + g'(u)^2}$$

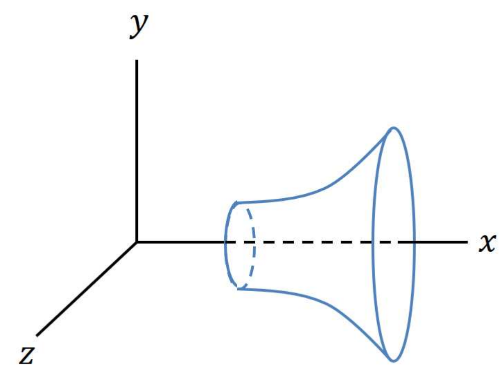
and so

$$A = \int\int_{\Omega} \|\vec{X}_1 \times \vec{X}_2\|\, du\, dv \qquad \text{(see page 37)}$$

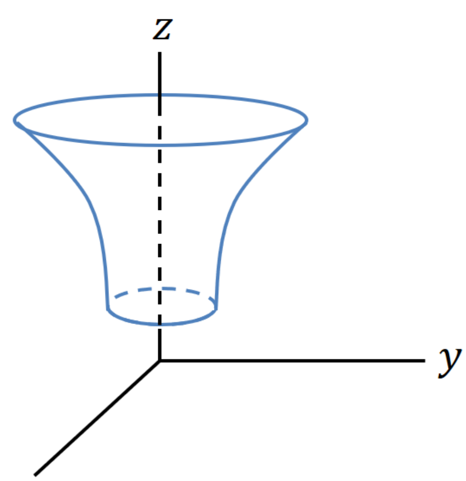
$$= \int_a^b \int_0^{2\pi} |f(u)|\sqrt{f'(u)^2+g'(u)^2}\,dv\,du$$
$$= 2\pi \int_a^b |f(u)|\sqrt{f'(u)^2+g'(u)^2}\,du.$$

(b) If $y = f(x)$, $x \in [a,b]$ where $a \geq 0$ is revolved about the $x-$axis, then we have:





This is equivalent to taking $\vec{X}(u,v) = (f(u), 0, u)$ (that is, the curve $x = f(z)$ in the $xz-$plane) and revolving it about the $z-$axis):



Then by Exercise 1.3.1, the surface is $\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, u)$. So by part (a), the surface area is

$$A = 2\pi \int_a^b |f(u)| \sqrt{f'(u)^2 + 1}\, du = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\, dx.$$

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