1.4 The First Fundamental Form

Note. Suppose $M$ is a surface determined by $\vec{X}(u,v) \subset E^3$ and suppose $\vec{a}(t)$ is a curve on $M$, $t \in [a,b]$. Then we can write $\vec{a}(t) = \vec{X}(u(t),v(t))$ (then $(u(t),v(t))$ is a curve in $\mathbb{R}^2$ whose image under $\vec{X}$ is $\vec{a}$). Then

$$\vec{a}'(t) = \frac{\partial \vec{X}}{\partial u} \frac{du}{dt} + \frac{\partial \vec{X}}{\partial v} \frac{dv}{dt} = u' \vec{X}_1 + v' \vec{X}_2. \tag{1}$$

If $s(t)$ represents the arc length along $\vec{a}$ (with $s(a) = 0$) then

$$s(t) = \int_a^t \|\vec{a}'(r)\| dr$$

and

$$\frac{ds}{dt} = \|\vec{a}'(t)\| \tag{2}$$

so

$$\left(\frac{ds}{dt}\right)^2 = \|\vec{a}'(t)\|^2 = \vec{a}' \cdot \vec{a}' = (u' \vec{X}_1 + v' \vec{X}_2) \cdot (u' \vec{X}_1 + v' \vec{X}_2)$$

$$= u'^2(\vec{X}_1 \cdot \vec{X}_1) + 2u'v'(\vec{X}_1 \cdot \vec{X}_2) + v'^2(\vec{X}_2 \cdot \vec{X}_2). \tag{3}$$

Following Gauss’ notation (briefly) we denote

$$E = \vec{X}_1 \cdot \vec{X}_1, \quad F = \vec{X}_1 \cdot \vec{X}_2, \quad G = \vec{X}_2 \cdot \vec{X}_2$$

and have

$$\left(\frac{ds}{dt}\right)^2 = E \left(\frac{du}{dt}\right)^2 + 2F \left(\frac{du}{dt} \frac{dv}{dt}\right) + G \left(\frac{dv}{dt}\right)^2 \tag{4}$$

or in differential notation

$$ds^2 = E(du)^2 + 2F(du)(dv) + G(dv)^2. \tag{5}$$

Definition. Let $M$ be a surface determined by $\vec{X}(u,v)$. The first fundamental form (or more commonly metric form) of $M$ is $\left(\frac{ds}{dt}\right)^2$ or $(ds)^2$ as defined above.
Definition. A property of a surface which depends only on the metric form of the surface is an intrinsic property.

Note. The idea of an intrinsic property is that a “resident” of the surface can detect such a property without appealing to a “larger space” in which the surface is embedded. Certainly an inhabitant of a surface can measure distance within the surface.

Example 10, page 32. Consider the $xy-$plane described as $\vec{X}(u,v) = (u,v,0)$ where $u \in \mathbb{R}$ and $v \in \mathbb{R}$. Then $\vec{X}_1 = (1,0,0)$ and $\vec{X}_2 = (0,1,0)$. So

$$ E = \vec{X}_1 \cdot \vec{X}_1 = 1, \quad F = \vec{X}_1 \cdot \vec{X}_2 = 0, \quad G = \vec{X}_2 \cdot \vec{X}_2 = 1. $$

Then the first fundamental form is

$$ \left( \frac{ds}{dt} \right)^2 = \left( \frac{du}{dt} \right)^2 + \left( \frac{dv}{dt} \right)^2 $$

or, in terms of $x$ and $y$:

$$ \left( \frac{ds}{dt} \right)^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2. $$

Of course, this is the “usual” expression for the differential of arclength from Calculus 2.

Definition. The matrix of the first fundamental form of a surface $M$ determined by $\vec{X}(u,v)$ is

$$ \begin{pmatrix} E & F \\ F & G \end{pmatrix} \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} $$

where $E$, $F$, $G$ are as defined as above.
Note. Since $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin \theta$ and $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$, where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$, then

$$\|\vec{X}_1 \times \vec{X}_2\|^2 = \|\vec{X}_1\|^2 \|\vec{X}_2\|^2 \sin^2 \theta = \|\vec{X}_1\|^2 \|\vec{X}_2\|^2 (1 - \cos^2 \theta)$$

$$= \|\vec{X}_1\|^2 \|\vec{X}_2\|^2 - (\|\vec{X}_1\| \|\vec{X}_2\| \cos \theta)^2$$

$$= (\vec{X}_1 \cdot \vec{X}_1) \cdot (\vec{X}_2 \cdot \vec{X}_2) - (\vec{X}_1 \cdot \vec{X}_2)^2$$

$$= EG - F^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix} = g.$$

Hence, $\|\vec{X}_1 \times \vec{X}_2\| = \sqrt{g}$.

Note. This matrix determines dot products of tangent vectors. If $\vec{v} = a\vec{X}_1 + b\vec{X}_2$ and $\vec{w} = c\vec{X}_1 + d\vec{X}_2$ are vectors tangent to a surface $M$ at a given point, then

$$\vec{v} \cdot \vec{w} = (a\vec{X}_1 + b\vec{X}_2) \cdot (c\vec{X}_1 + d\vec{X}_2) = Eac + F(ad + bc) + Gbd$$

$$= (a, b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

Notation. We now replace the parameters $u$ and $v$ with $u^1$ and $u^2$. We then have

$$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2 = \sum_{i,j} g_{ij}du^i du^j$$

where the summation is taken (throughout this chapter) over the set $\{1, 2\}$. In Chapter 3, we will sum over $\{1, 2, 3, 4\}$. If $\vec{v}$ is a vector tangent to $M$ at a point $\vec{P}$ and $\vec{v} = (v^1, v^2)$ in the basis $\{\vec{X}_1, \vec{X}_2\}$ for the tangent plane at $\vec{P}$, then we have

$$\vec{v} = \sum_i v^i \vec{X}_i.$$
If $\vec{\alpha}(t)$ is a curve on $M$ where $\vec{\alpha}$ is represented by $\vec{X}(u^1(t), u^2(t))$ then

$$\vec{\alpha}'(t) = u^1'(t)\vec{X}_1 + u^2'(t)\vec{X}_2 = \sum_i u^i' \vec{X}_i.$$ 

**Notation.** We denote the $ij$ entry of $(g_{ij})^{-1}$ as $g^{ij}$. Therefore $(g_{ij})(g^{ij}) = I$ and

$$\sum_j g_{ij} g^{jk} = \delta^k_i \text{ (the } ik\text{ entry of } I)$$

where

$$\delta^k_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

**Example (Exercise 1.4.3(c)).** For the surface $\vec{X}(u, v) = (u \cos v, u \sin v, bv)$ (the helicoid of Example 9), compute the matrix $(g_{ij})$, its determinate $g$, the inverse matrix $(g^{ij})$ and the unit normal vector $\vec{U}$.

From Wikipedia’s page

https://commons.wikimedia.org/wiki/File:Helicoid_JD.png
Solution. Well

\[ \vec{X}_1 = \frac{\partial \vec{X}}{\partial u} = (\cos v, \sin v, 0) \]
\[ \vec{X}_2 = \frac{\partial \vec{X}}{\partial v} = (-u \sin v, u \cos v, b) \]

and so

\[
g_{11} = \vec{X}_1 \cdot \vec{X}_1 = \cos^2 v + \sin^2 v + 0 = 1
\]
\[
g_{22} = \vec{X}_2 \cdot \vec{X}_2 = u^2 \sin^2 v + u^2 \cos^2 v + b^2 = u^2 + b^2
\]
\[
g_{12} = \vec{X}_1 \cdot \vec{X}_2 = -u \cos v \sin v + u \cos v \sin v + 0 = 0 = g_{21}.
\]

Therefore

\[
G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + b^2 \end{pmatrix}
\]

and \( g = \det(g_{ij}) = u^2 + b^2 \). Then

\[
G^{-1} = \begin{pmatrix} g_{11}^{12} & g_{12}^{12} \\ g_{21}^{22} & g_{22}^{22} \end{pmatrix} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}
\]
\[
= \frac{1}{u^2 + b^2} \begin{pmatrix} u^2 + b^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2 + b^2} \end{pmatrix}.
\]

Now the unit normal vector is \( \vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\| \vec{X}_1 \times \vec{X}_2 \|} \) and

\[
\vec{X}_1 \times \vec{X}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & b \end{vmatrix}
\]
\[
= (b \sin v, -b \cos v, u \cos^2 v + u \sin^2 v) = (b \sin v, -b \cos v, u).
\]

Now

\[
\| \vec{X}_1 \times \vec{X}_2 \| = \sqrt{b^2 \sin^2 v + b^2 \cos^2 v + u^2} = \sqrt{b^2 + u^2}.
\]
Therefore
\[
\vec{U} = \left( \frac{b \sin v}{\sqrt{b^2 + u^2}}, \frac{-b \cos v}{\sqrt{b^2 + u^2}}, \frac{u}{\sqrt{b^2 + u^2}} \right).
\]

**Definition.** Suppose \( \Omega \) is a closed subset of the \( u^1u^2 \)-plane and that \( \vec{X} : \Omega \to E^3 \) is smooth (i.e. has continuous first partials), is one-to-one and regular (i.e. \( \vec{X}_1 \) and \( \vec{X}_2 \) are linearly independent) on the interior of \( \Omega \). Then the *area* of the surface \( \vec{X}(\Omega) \) is
\[
A = \int \int_{\Omega} \|\vec{X}_1 \times \vec{X}_2\| du^1 du^2 = \int \int_{\Omega} \sqrt{g} du^1 du^2.
\]
(See page 37 of the text for motivation of this definition.)

**Example (Exercise 1.4.6).** (a) Show that the area \( A \) of the surface of revolution \( \vec{X}(u,v) = (f(u) \cos v, f(u) \sin v, g(u)) \) where \( u \in [a,b] \) and \( v \in [0,2\pi] \) is given by
\[
A = 2\pi \int_a^b |f(u)| \sqrt{f'(u)^2 + g'(u)^2} \, du.
\]
(b) Show that the area of the surface obtained by revolving the graph \( y = f(x) \) for \( x \in [a,b] \) about the \( x \)-axis is given by
\[
A = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2} \, dx.
\]

**Solution.** (a) Consider the surface area of the surface of revolution \( \vec{X}(u,v) = (f(u) \cos v, f(u) \sin v, g(u)) \). We have (from Exercise 1.4.5)
\[
\|\vec{X}_1 \times \vec{X}_2\| = |f(u)| \sqrt{f'(u)^2 + g'(u)^2}
\]
and so
\[
A = \int \int_{\Omega} \|\vec{X}_1 \times \vec{X}_2\| \, du \, dv \quad \text{(see page 37)}
\]
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\[
\int_a^b \int_0^{2\pi} |f(u)| \sqrt{f'(u)^2 + g'(u)^2} \, dv \, du = 2\pi \int_a^b |f(u)| \sqrt{f'(u)^2 + g'(u)^2} \, du.
\]

(b) If \( y = f(x) \), \( x \in [a, b] \) where \( a \geq 0 \) is revolved about the \( x \)-axis, then we have:

This is equivalent to taking \( \vec{X}(u, v) = (f(u), 0, u) \) (that is, the curve \( x = f(z) \) in the \( xz \)-plane) and revolving it about the \( z \)-axis):

Then by Exercise 1.3.1, the surface is \( \vec{X}(u, v) = (f(u) \cos v, f(u) \sin v, u) \). So by
part (a), the surface area is

\[ A = 2\pi \int_{a}^{b} |f(u)| \sqrt{f'(u)^2 + 1} \, du = 2\pi \int_{a}^{b} |f(x)| \sqrt{1 + f'(x)^2} \, dx. \]

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