1.5 The Second Fundamental Form

Notation. We adopt the *Einstein summation convention* in which any expression that has a single index appearing both as a subscript and a superscript is assumed to be summed over that index.

Example. We denote $\sum_i v^i \vec{X}_i$ as $v^i \vec{X}_i$.

Example. We denote $\sum_{i,j} g_{ij} v^i w^j$ as $g_{ij} v^i w^j$.

Note. We have treated a path $\vec{\alpha}(t)$ along a surface M as if it were the trajectory of a particle in E^3. We then interprete $\vec{\alpha}''(t)$ as the acceleration of the particle. Well, a particle can accelerate in two different ways: (1) it can accelerate in the direction of travel, and (2) it can accelerate by changing its direction of travel. We can therefore decompose $\vec{\alpha}''$ into two components, $\vec{\alpha}''_{\vec{T}}$ (representing acceleration in the direction of travel) and $\vec{\alpha}''_{\vec{N}}$ (representing acceleration that changes the direction of travel). You may have dealt with this in Calculus 3 by taking $\vec{\alpha}''_{\vec{T}}$ as the component of $\vec{\alpha}''$ in the direction of $\vec{\alpha}'$, computed as

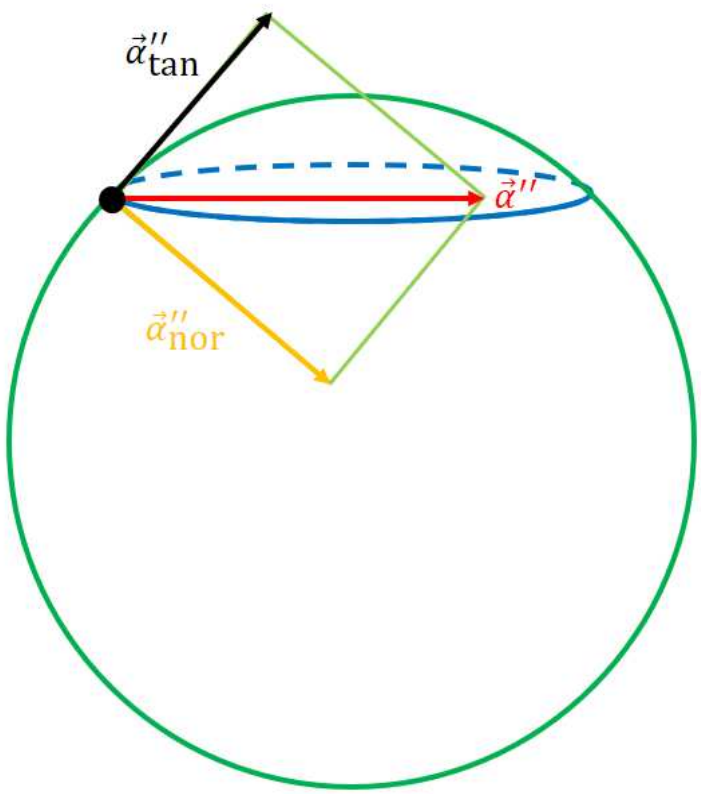
$$\vec{\alpha}''_{\vec{T}} = \left(\vec{\alpha}'' \cdot \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}\right) \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}$$

and $\vec{\alpha}''_{\vec{N}}$ as the "remaining component" of $\vec{\alpha}$ (that is, $\vec{\alpha}''_{\vec{N}} = \vec{\alpha}'' - \vec{\alpha}''_{\vec{T}}$). This is reminiscent of the *Frenet formulas* or the *Frenet frame* $(\vec{T}, \vec{N}, \vec{B})$ of Exercise 1.1.14).

Note. With $\vec{\alpha}$ parameterized in terms of arc length s, $\vec{\alpha} = \vec{\alpha}(s) = \vec{X}(u^1(s), u^2(s))$ we have the unit tangent vector $\vec{T}(s) = \vec{\alpha}'(s) = u^{i\prime}\vec{X}_i$. We saw in Section 1.1 that $\vec{\alpha}''(s) = \vec{T}'(s)$ is a vector normal to $\vec{\alpha}'$ ($\vec{T}' = k\vec{N}$ - see Exercise 1.1.14). <u>In this section</u>, we again decompose $\vec{\alpha}''$ into two orthogonal components, <u>but this time</u> we make explicit use of the surface M. We wish to write

$$\vec{\alpha}'' = \vec{\alpha}''_{\text{tan}} + \vec{\alpha}''_{\text{nor}}$$

where $\vec{\alpha}''_{\text{tan}}$ is the component of $\vec{\alpha}''$ tangent to M and $\vec{\alpha}''_{\text{nor}}$ is the component of $\vec{\alpha}''$ normal to M. Notice that $\vec{\alpha}''_{\text{tan}}$ will be a linear combination of $\vec{X}_1$ and $\vec{X}_2$ (they are a basis for the tangent plane, recall) and $\vec{\alpha}''_{\text{nor}}$ will be a multiple of the unit normal vector to M, $\vec{U}$ (calculated as $\vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}$).



In the figure, $\vec{\alpha}''$, $\vec{\alpha}''_{\text{tan}}$, and $\vec{\alpha}''_{\text{nor}}$ all line in the same plane.

Note. Since $\vec{\alpha}(s) = \vec{X}(u^1(s), u^2(s))$ and $\vec{\alpha}' = u^{i\prime}\vec{X}_i$ (here, $\prime$ means d/ds), then

$$\vec{\alpha}'' = u^{i\prime\prime}\vec{X}_i + u^{i\prime}\vec{X}_i' = u^{i\prime\prime}\vec{X}_i + u^{i\prime}\frac{d\vec{X}_i}{ds}.$$

Now $u^{i\prime\prime}\vec{X}_i$ is *part* of $\vec{\alpha}''_{\text{tan}}$, but $u^{i\prime}\vec{X}_i'$ may also have a component in the tangent plane. Well,

$$\begin{aligned}\frac{d\vec{X}_i}{ds} &= \frac{d}{ds}\left[\vec{X}_i(u^1(s), u^2(s))\right] = \frac{\partial \vec{X}_i}{\partial u^1}\frac{du^1}{ds} + \frac{\partial \vec{X}_i}{\partial u^2}\frac{du^2}{ds} \\ &= \frac{\partial \vec{X}_i}{\partial u^1}u^{1\prime} + \frac{\partial \vec{X}_i}{\partial u^2}u^{2\prime}\end{aligned}$$

(notice that we sum over j in the last term, since we treat the partial derivative with respect to u^j as if j were a subscript). If we denote $\dfrac{\partial^2 \vec{X}}{\partial u^i \partial u^j} = \vec{X}_{ij}$ (we have assumed continuous second partials, so the order of differentiation doesn't matter) then we have $\dfrac{d\vec{X}_i}{ds} = \vec{X}_{ij}u^{j\prime}$. So acceleration becomes

$$\vec{\alpha}'' = u^{r\prime\prime}\vec{X}_r + u^{i\prime}u^{j\prime}\vec{X}_{ij}.$$

We now need only to write $\vec{X}_{ij}$ in terms of a component in the tangent plane (and so in terms of $\vec{X}_1$ and $\vec{X}_2$) and a component normal to the tangent plane (which will be a multiple of $\vec{U}$).

Definition. With the notation above, we define the *formulae of Gauss* as

$$\vec{X}_{ij} = \Gamma^r_{ij}\vec{X}_r + L_{ij}\vec{U}.$$

That is we define L_{ij} as the projection of $\vec{X}_{ij}$ in the direction $\vec{U}$. Notice, however, that Γ^r_{ij} may not be the projection of $\vec{X}_{ij}$ onto $\vec{X}_r$ since the $\vec{X}_r$'s are not orthonormal.

Note. Since projections are computed from dot products, we immediately have that

$$L_{ij} = \vec{X}_{ij} \cdot \vec{U} = \vec{X}_{ij} \cdot \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}.$$

Note. We therefore have

$$\vec{\alpha}'' = \vec{\alpha}''_{\text{tan}} + \vec{\alpha}''_{\text{nor}} = \left(u^{r\prime\prime} + \Gamma^r_{ij} u^{i\prime} u^{j\prime}\right) \vec{X}_r + \left(L_{ij} u^{i\prime} u^{j\prime}\right) \vec{U}.$$

Definition. The *second fundamental form* of surface M is the matrix

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

(Notice this differs from the text's definition on page 44.) We denote the determinate of this matrix as L.

Note. The second fundamental form is a function of $u = u^1$ and $v = u^2$. Also, since we have $\vec{X}_{12} = \vec{X}_{21}$, it follows that $L_{12} = L_{21}$ and so (L_{ij}) is a symmetric matrix.

Note. We will see that the second fundamental form reflects the *extrinsic* geometry of surface M (that is, the way M is embedded in E^3; "how it curves relative to that space" as the text says).

Example (Exercise 1.5.2). Compute L from the second fundamental form of the surface of revolution

$$\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

Solution. Well

$$\begin{aligned} \vec{X}_1 &= \frac{\partial \vec{X}}{\partial u} = (f'(u)\cos v, f'(u)\sin v, g'(u)) \\ \vec{X}_2 &= \frac{\partial \vec{X}}{\partial v} = (-f(u)\sin v, f(u)\cos v, 0) \end{aligned}$$

and so (from Exercise 1.4.5)

$$\vec{U} = \frac{f(u)}{|f(u)|\sqrt{f'(u)^2 + g'(u)^2}}(-g'(u)\cos v, -g'(u)\sin v, f'(u)).$$

Next,

$$\begin{aligned} \vec{X}_{11} &= \frac{\partial^2 \vec{X}}{\partial^2 u} = (f''(u)\cos v, f''(u)\sin v, g''(u)) \\ \vec{X}_{22} &= \frac{\partial^2 \vec{X}}{\partial^2 v} = (-f(u)\cos v, -f(u)\sin v, 0) \\ \vec{X}_{12} &= \frac{\partial^2 \vec{X}}{\partial u \partial v} = (-f'(u)\sin v, f'(u)\cos v, 0) = \vec{X}_{21}. \end{aligned}$$

So

$$\begin{aligned} L_{11} &= \vec{X}_{11} \cdot \vec{U} = \frac{f(u)}{|f(u)|\sqrt{f'(u)^2 + g'(u)^2}}(-f''(u)g'(u)\cos^2 v \\ &\quad -f''(u)g'(u)\sin^2 v + f'(u)g''(u)) \\ &= \frac{f(u)(f'(u)g''(u) - f''(u)g'(u))}{|f(u)|\sqrt{f'(u)^2 + g'(u)^2}} \\ L_{12} &= \vec{X}_{12} \cdot \vec{U} = \frac{(f(u)g'(u)\cos v \sin v - f(u)g'(u)\cos v \sin v + 0)f(u)}{|f(u)|\sqrt{f'(u)^2 + g'(u)^2}} \\ &\quad = 0 = L_{21} \end{aligned}$$

$$\begin{aligned} L_{22} &= \vec{X}_{22} \cdot \vec{U} = \frac{f(u)}{|f(u)|\sqrt{f'(u)^2 + g'(u)}}(f(u)g'(u)\cos^2 v \\ &\quad + f(u)g'(u)\sin^2 v + 0) \\ &= \frac{f(u)^2 g'(u)}{|f(u)|\sqrt{f'(u) + g'(u)^2}} = \frac{|f(u)|g'(u)}{\sqrt{f'(u)^2 + g'(u)^2}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} L &= \det L_{ij} = L_{11}L_{22} - L_{12}L_{21} \\ &= \frac{f(u)(f'(u)g''(u) - f''(u)g'(u))}{|f(u)|\sqrt{f'(u)^2 + g'(u)^2}} \frac{|f(u)|g'(u)}{\sqrt{f'(u)^2 + g'(u)^2}} \\ &= \frac{f(u)g'(u)(f'(u)g''(u) - f''(u)g'(u))}{f'(u)^2 + g'(u)^2}. \end{aligned}$$

Definition. Let $\vec{v} = v^i \vec{X}_i$ be a unit vector tangent to M at $\vec{P}$. The *normal curvature of M at $\vec{P}$ in the direction $\vec{v}$*, denoted $k_n(\vec{v})$ is

$$k_n(\vec{v}) = L_{ij} v^i v^j$$

where $\vec{v} = (v^1, v^2)$ (the coordinate vector of $\vec{v}$ with respect to the ordered bases $(\vec{X}_1, \vec{X}_2)$ of the tangent plane).

Example (Exercise 1.5.5). Find the normal curvature of the surface $z = f(x, y)$ at an arbitrary point, in the direction of a unit tangent vector (a, b, c) at that point.

Solution. We have

$$\begin{aligned} \vec{X}_1 &= \frac{\partial \vec{X}}{\partial u} = (1, 0, \frac{\partial f}{\partial u}(u, v)) = (1, 0, f_u) \\ \vec{X}_2 &= \frac{\partial \vec{X}}{\partial v} = (0, 1, \frac{\partial f}{\partial v}(u, v)) = (0, 1, f_v). \end{aligned}$$

So

$$\vec{X}_1 \times \vec{X}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)$$

and $\|\vec{X}_1 \times \vec{X}_2\| = \sqrt{(f_u)^2 + (f_v)^2 + 1}$. Therefore

$$\vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} = \frac{1}{\sqrt{(f_u)^2 + (f_v)^2 + 1}}(-f_u, -f_v, 1).$$

Now

$$\begin{aligned} \vec{X}_{11} &= \frac{\partial^2 \vec{X}}{\partial u^2} = (0, 0, f_{uu}) \\ \vec{X}_{12} &= \frac{\partial^2 \vec{X}}{\partial u\, \partial v} = (0, 0, f_{uv}) = \vec{X}_{21} \\ \vec{X}_{22} &= \frac{\partial^2 \vec{X}}{\partial v^2} = (0, 0, f_{vv}) \end{aligned}$$

and so

$$\begin{aligned} L_{11} &= \vec{X}_{11} \cdot \vec{U} = \frac{f_{uu}}{\sqrt{(f_u)^2 + (f_v)^2 + 1}} \\ L_{22} &= \vec{X}_{22} \cdot \vec{U} = \frac{f_{vv}}{\sqrt{(f_u)^2 + (f_v)^2 + 1}} \\ L_{12} &= \vec{X}_{12} \cdot \vec{U} = \frac{f_{uv}}{\sqrt{(f_u)^2 + (f_v)^2 + 1}} = L_{21}. \end{aligned}$$

Now $\vec{v} = v^i \vec{X}_i = v^1(1, 0, f_u) + v^2(0, 1, f_v) = (a, b, c)$, implying that $v^1 = a$ and $v^2 = b$. Hence

$$\begin{aligned} k_n(\vec{v}) &= L_{ij} v^i v^j \\ &= L_{11} v^1 v^1 + 2L_{12} v^1 v^2 + L_{22} v^2 v^2 \\ &= \frac{1}{\sqrt{(f_u)^2 + (f_v)^2 + 1}}(a^2 f_{uu} + 2ab f_{uv} + b^2 f_{vv}). \end{aligned}$$

Note. If $\vec{\alpha} = \vec{X}(u^1(s), u^2(s))$ is a curve on M, $\vec{P}$ is a point on M with $\vec{\alpha}(s_0) = \vec{P}$ and $\vec{v} = \vec{\alpha}'(s_0)$ then $\vec{\alpha}'(s_0) = u^{i\prime}(s_0)\vec{X}_i(u^1(s_0), u^2(s_0))$ and so $v^i = u^{i\prime}(s_0)$ (see page 35 for representation of a tangent vector: $\vec{v} = v^i\vec{X}_i$). Therefore

$$k_n(\vec{v}) = L_{ij}v^iv^j = L_{ij}u^{i\prime}u^{j\prime}.$$

Now $\vec{\alpha}'' = u^{r\prime\prime}\vec{X}_r + u^{i\prime}u^{j\prime}\vec{X}_{ij}$ (equation (16), page 43), and $\vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}$ so

$$\begin{aligned}
\vec{\alpha}'' \cdot \vec{U} &= \left(u^{r\prime\prime}\vec{X}_r + u^{i\prime}u^{j\prime}\vec{X}_{ij}\right) \cdot \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} \\
&= \vec{0} + u^{i\prime}u^{j\prime}\left(\vec{X}_{ij} \cdot \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}\right) = u^{i\prime}u^{j\prime}L_{ij}.
\end{aligned}$$

Hence $k_n(\vec{v}) = \vec{\alpha}'' \cdot \vec{U}$. This equation is used in Exercise 1.5.6.

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