1.5 The Second Fundamental Form

**Notation.** We adopt the *Einstein summation convention* in which any expression that has a single index appearing both as a subscript and a superscript is assumed to be summed over that index.

**Example.** We denote \( \sum_i v^i \vec{X}_i \) as \( v^i \vec{X}_i \).

**Example.** We denote \( \sum_{i,j} g_{ij} v^i w^j \) as \( g_{ij} v^i w^j \).

**Note.** We have treated a path \( \vec{\alpha}(t) \) along a surface \( M \) as if it were the trajectory of a particle in \( E^3 \). We then interprete \( \vec{\alpha}''(t) \) as the acceleration of the particle. Well, a particle can accelerate in two different ways: (1) it can accelerate in the direction of travel, and (2) it can accelerate by changing its direction of travel. We can therefore decompose \( \vec{\alpha}'' \) into two components, \( \vec{\alpha}''_T \) (representing acceleration in the direction of travel) and \( \vec{\alpha}''_N \) (representing acceleration that changes the direction of travel). You may have dealt with this in Calculus 3 by taking \( \vec{\alpha}''_T \) as the component of \( \vec{\alpha}'' \) in the direction of \( \vec{\alpha}' \), computed as

\[
\vec{\alpha}''_T = \left( \vec{\alpha}'' \cdot \frac{\vec{\alpha}'}{||\vec{\alpha}'||} \right) \frac{\vec{\alpha}'}{||\vec{\alpha}'||}
\]

and \( \vec{\alpha}''_N \) as the “remaining component” of \( \vec{\alpha} \) (that is, \( \vec{\alpha}''_N = \vec{\alpha}'' - \vec{\alpha}''_T \)). This is reminiscent of the *Frenet formulas* or the *Frenet frame* \((\vec{T}, \vec{N}, \vec{B})\) of Exercise 1.1.14).
Note. With $\bar{\alpha}$ parameterized in terms of arc length $s$, $\bar{\alpha} = \bar{\alpha}(s) = \bar{X}(u^1(s), u^2(s))$ we have the unit tangent vector $\bar{T}(s) = \bar{\alpha}'(s) = u^i \bar{X}_i$. We saw in Section 1.1 that $\bar{\alpha}''(s) = \bar{T}'(s)$ is a vector normal to $\bar{\alpha}'$ ($\bar{T}' = k \bar{N}$ - see Exercise 1.1.14). In this section, we again decompose $\bar{\alpha}''$ into two orthogonal components, but this time we make explicit use of the surface $M$. We wish to write

$$\bar{\alpha}'' = \bar{\alpha}''_{\text{tan}} + \bar{\alpha}''_{\text{nor}}$$

where $\bar{\alpha}''_{\text{tan}}$ is the component of $\bar{\alpha}''$ tangent to $M$ and $\bar{\alpha}''_{\text{nor}}$ is the component of $\bar{\alpha}''$ normal to $M$. Notice that $\bar{\alpha}''_{\text{tan}}$ will be a linear combination of $\bar{X}_1$ and $\bar{X}_2$ (they are a basis for the tangent plane, recall) and $\bar{\alpha}''_{\text{nor}}$ will be a multiple of the unit normal vector to $M$, $\bar{U}$ (calculated as $\bar{U} = \frac{\bar{X}_1 \times \bar{X}_2}{\|\bar{X}_1 \times \bar{X}_2\|}$).

In the figure, $\bar{\alpha}''$, $\bar{\alpha}''_{\text{tan}}$, and $\bar{\alpha}''_{\text{nor}}$ all line in the same plane.
Note. Since $\vec{\alpha}(s) = \vec{X}(u^1(s), u^2(s))$ and $\vec{\alpha}' = u^r \vec{X}_r$ (here, $r$ means $d/ds$), then

$$\vec{\alpha}'' = u'' \vec{X}_i + u'^r \vec{X}'_i = u'' \vec{X}_i + u'^r \frac{d \vec{X}_i}{ds}.$$ 

Now $u'' \vec{X}_i$ is part of $\vec{\alpha}''_{\text{tan}}$, but $u'^r \vec{X}'_i$ may also have a component in the tangent plane. Well,

$$\frac{d \vec{X}_i}{ds} = \frac{d}{ds} \left[ \vec{X}_i(u^1(s), u^2(s)) \right] = \frac{\partial \vec{X}_i}{\partial u^1} \frac{du^1}{ds} + \frac{\partial \vec{X}_i}{\partial u^2} \frac{du^2}{ds}$$

$$= \frac{\partial \vec{X}_i}{\partial u^1} u'^1 + \frac{\partial \vec{X}_i}{\partial u^2} u'^2$$

(notice that we sum over $j$ in the last term, since we treat the partial derivative with respect to $u^j$ as if $j$ were a subscript). If we denote $\frac{\partial^2 \vec{X}}{\partial u^i \partial u^j} = \vec{X}_{ij}$ (we have assumed continuous second partials, so the order of differentiation doesn’t matter) then we have $\frac{d \vec{X}_i}{ds} = \vec{X}_{ij} u'^j$. So acceleration becomes

$$\vec{\alpha}'' = u'' \vec{X}_r + u'^r u'^j \vec{X}_{ij}.$$ 

We now need only to write $\vec{X}_{ij}$ in terms of a component in the tangent plane (and so in terms of $\vec{X}_1$ and $\vec{X}_2$) and a component normal to the tangent plane (which will be a multiple of $\vec{U}$).

Definition. With the notation above, we define the formulae of Gauss as

$$\vec{X}_{ij} = \Gamma^r_{ij} \vec{X}_r + L_{ij} \vec{U}.$$ 

That is we define $L_{ij}$ as the projection of $\vec{X}_{ij}$ in the direction $\vec{U}$. Notice, however, that $\Gamma^r_{ij}$ may not be the projection of $\vec{X}_{ij}$ onto $\vec{X}_r$ since the $\vec{X}_r$'s are not orthonormal.
Note. Since projections are computed from dot products, we immediately have that
\[ L_{ij} = \tilde{X}_{ij} \cdot \tilde{U} = \tilde{X}_{ij} \cdot \frac{\tilde{X}_1 \times \tilde{X}_2}{\|\tilde{X}_1 \times \tilde{X}_2\|}. \]

Note. We therefore have
\[ \tilde{\alpha}'' = \tilde{\alpha}''_{\tan} + \tilde{\alpha}''_{\norn} = (u''' + \Gamma^r_{ij} u^{i'} u^{j'}) \tilde{X}_r + (L_{ij} u^{i'} u^{j'}) \tilde{U}. \]

Definition. The second fundamental form of surface \( M \) is the matrix
\[ \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}. \]
(Notice this differs from the text’s definition on page 44.) We denote the determinant of this matrix as \( L \).

Note. The second fundamental form is a function of \( u = u^1 \) and \( v = u^2 \). Also, since we have \( \tilde{X}_{12} = \tilde{X}_{21} \), it follows that \( L_{12} = L_{21} \) and so \( (L_{ij}) \) is a symmetric matrix.

Note. We will see that the second fundamental form reflects the extrinsic geometry of surface \( M \) (that is, the way \( M \) is embedded in \( E^3 \); “how it curves relative to that space” as the text says).
Example (Exercise 1.5.2). Compute $L$ from the second fundamental form of the surface of revolution

$$\vec{X}(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$

Solution. Well

$$\vec{X}_1 = \frac{\partial \vec{X}}{\partial u} = (f'(u) \cos v, f'(u) \sin v, g'(u))$$

$$\vec{X}_2 = \frac{\partial \vec{X}}{\partial v} = (-f(u) \sin v, f(u) \cos v, 0)$$

and so (from Exercise 1.4.5)

$$\vec{U} = \frac{f(u)}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}} (-g'(u) \cos v, -g'(u) \sin v, f'(u)).$$

Next,

$$\vec{X}_{11} = \frac{\partial^2 \vec{X}}{\partial^2 u} = (f''(u) \cos v, f''(u) \sin v, g''(u))$$

$$\vec{X}_{22} = \frac{\partial^2 \vec{X}}{\partial^2 v} = (-f(u) \cos v, -f(u) \sin v, 0)$$

$$\vec{X}_{12} = \frac{\partial^2 \vec{X}}{\partial u \partial v} = (-f'(u) \sin v, f'(u) \cos v, 0) = \vec{X}_{21}.$$

So

$$L_{11} = \vec{X}_{11} \cdot \vec{U} = \frac{f(u)}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}} (-f''(u)g'(u) \cos^2 v$$

$$-f''(u)g'(u) \sin^2 v + f'(u)g''(u))$$

$$= \frac{f(u)(f'(u)g''(u) - f''(u)g'(u))}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}}$$

$$L_{12} = \vec{X}_{12} \cdot \vec{U} = \frac{(f(u)g'(u) \cos v \sin v - f(u)g'(u) \cos v \sin v + 0)f(u)}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}}$$

$$= 0 = L_{21}.$$
\[ L_{22} = \vec{X}_{22} \cdot \vec{U} = \frac{f(u)}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}} (f(u)g'(u) \cos^2 v + f(u)g'(u) \sin^2 v + 0) \]
\[ = \frac{f(u)^2 g'(u)}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}} = \frac{|f(u)| g'(u)}{\sqrt{f'(u)^2 + g'(u)^2}}. \]

Therefore we have
\[ L = \det L_{ij} = L_{11}L_{22} - L_{12}L_{21} \]
\[ = \frac{f(u)(f'(u)g''(u) - f''(u)g'(u))}{|f(u)| \sqrt{f'(u)^2 + g'(u)^2}} \frac{|f(u)| g'(u)}{\sqrt{f'(u)^2 + g'(u)^2}} \]
\[ = \frac{f(u)g'(u)(f'(u)g''(u) - f''(u)g'(u))}{f'(u)^2 + g'(u)^2}. \]

**Definition.** Let \( \vec{v} = v^i \vec{X}_i \) be a unit vector tangent to \( M \) at \( \vec{P} \). The *normal curvature of \( M \) at \( \vec{P} \) in the direction \( \vec{v} \), denoted \( k_n(\vec{v}) \) is
\[ k_n(\vec{v}) = L_{ij} v^i v^j \]
where \( \vec{v} = (v^1, v^2) \) (the coordinate vector of \( \vec{v} \) with respect to the ordered bases \( (\vec{X}_1, \vec{X}_2) \) of the tangent plane).

**Example (Exercise 1.5.5).** Find the normal curvature of the surface \( z = f(x, y) \) at an arbitrary point, in the direction of a unit tangent vector \( (a, b, c) \) at that point.

**Solution.** We have
\[ \vec{X}_1 = \frac{\partial \vec{X}}{\partial u} = (1, 0, \frac{\partial f}{\partial u}(u, v)) = (1, 0, f_u) \]
\[ \vec{X}_2 = \frac{\partial \vec{X}}{\partial v} = (0, 1, \frac{\partial f}{\partial v}(u, v)) = (0, 1, f_v). \]
So
\[
\vec{X}_1 \times \vec{X}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)
\]
and \(\|\vec{X}_1 \times \vec{X}_2\| = \sqrt{(f_u)^2 + (f_v)^2 + 1}\). Therefore
\[
\vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} = \frac{1}{\sqrt{(f_u)^2 + (f_v)^2 + 1}}(-f_u, -f_v, 1).
\]
Now
\[
\begin{align*}
\vec{X}_{11} &= \frac{\partial^2 \vec{X}}{\partial u^2} = (0, 0, f_{uu}) \\
\vec{X}_{12} &= \frac{\partial^2 \vec{X}}{\partial u \partial v} = (0, 0, f_{uv}) = \vec{X}_{21} \\
\vec{X}_{22} &= \frac{\partial^2 \vec{X}}{\partial v^2} = (0, 0, f_{vv})
\end{align*}
\]
and so
\[
\begin{align*}
L_{11} &= \vec{X}_{11} \cdot \vec{U} = \frac{f_{uu}}{\sqrt{(f_u)^2 + (f_v)^2 + 1}} \\
L_{22} &= \vec{X}_{22} \cdot \vec{U} = \frac{f_{vv}}{\sqrt{(f_u)^2 + (f_v)^2 + 1}} \\
L_{12} &= \vec{X}_{12} \cdot \vec{U} = \frac{f_{uv}}{\sqrt{(f_u)^2 + (f_v)^2 + 1}} = L_{21}.
\end{align*}
\]
Now \(\vec{v} = v^i \vec{X}_i = v^1(1, 0, f_u) + v^2(0, 1, f_v) = (a, b, c)\), implying that \(v^1 = a\) and \(v^2 = b\). Hence
\[
\begin{align*}
k_n(\vec{v}) &= L_{ij} v^i v^j \\
&= L_{11} v^1 v^1 + 2L_{12} v^1 v^2 + L_{22} v^2 v^2 \\
&= \frac{1}{\sqrt{(f_u)^2 + (f_v)^2 + 1}}(a^2 f_{uu} + 2ab f_{uv} + b^2 f_{vv}).
\end{align*}
\]
Note. If $\vec{x} = \vec{X}(u^1(s), u^2(s))$ is a curve on $M$, $\vec{P}$ is a point on $M$ with $\vec{x}(s_0) = \vec{P}$ and $\vec{v} = \vec{x}'(s_0)$ then $\vec{x}'(s_0) = u''(s_0)\vec{X}_i(u^1(s_0), u^2(s_0))$ and so $v^i = u''(s_0)$ (see page 35 for representation of a tangent vector: $\vec{v} = v^i \vec{X}_i$). Therefore

$$k_n(\vec{v}) = L_{ij} v^i v^j = L_{ij} u'' u''.$$

Now $\vec{x}'' = u'' \vec{X}_r + u' u'' \vec{X}_{ij}$ (equation (16), page 43), and $\vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\| \vec{X}_1 \times \vec{X}_2 \|}$ so

$$\vec{x}'' \cdot \vec{U} = \left( u'' \vec{X}_r + u' u'' \vec{X}_{ij} \right) \cdot \frac{\vec{X}_1 \times \vec{X}_2}{\| \vec{X}_1 \times \vec{X}_2 \|}$$

$$= 0 + u' u'' \left( \vec{X}_{ij} \cdot \frac{\vec{X}_1 \times \vec{X}_2}{\| \vec{X}_1 \times \vec{X}_2 \|} \right) = u' u'' L_{ij}.$$

Hence $k_n(\vec{v}) = \vec{x}'' \cdot \vec{U}$. This equation is used in Exercise 1.5.6.

Revised: 6/13/2019