1.6 The Gauss Curvature in Detail

Note. We have defined the normal curvature of a surface at a point $\vec{P}$ in the direction $\vec{v}$: $k_n(\vec{v})$. Therefore, for a given point on a surface, there are an infinite number of (not necessarily distinct) curvatures (one for each "direction"). We can think of $k_n(\vec{v})$ as a function mapping the vector space $T_{\vec{P}}(M)$ (the plane tangent to surface M at point $\vec{P}$) into $\mathbb{R}$. That is $k_n : T_{\vec{P}}(M) \to \mathbb{R}$. We need $\vec{v}$ to be a unit vector, so the domain of k_n is $\{\vec{v} \in T_{\vec{P}}(M) \mid \|\vec{v}\| = 1\}$. Therefore, k_n is a continuous function on a compact set and by the Extreme Value Theorem (for metric spaces), k_n assumes a maximum and a minimum value.

Definition. Let M be a surface and $\vec{P}$ a point on the surface. Define $k_1 = \max k_n(\vec{v})$ and $k_2 = \min k_n(\vec{v})$ where the maximum and minimum are taken over the domain of k_n. k_1 and k_2 are called the *principal curvatures* of M at $\vec{P}$, and the corresponding directions are called *principal directions*. The product $K = K(P) = k_1 k_2$ is the *Gauss curvature* of M at $\vec{P}$.

Theorem I-5. The Gauss curvature at any point $\vec{P}$ of a surface M is $K(\vec{P}) = L/g$ where $L = \det(L_{ij})$ and $g = \det(g_{ij})$.

Proof. First, if $\vec{v} = v^i \vec{X}_i$ then

$$\begin{aligned} \|\vec{v}\|^2 &= \left(v^1\vec{X}_1 + v^2\vec{X}_2\right) \cdot \left(v^1\vec{X}_1 + v^2\vec{X}_2\right) \\ &= (v^1)^2\vec{X}_1 \cdot \vec{X}_1 + 2(v^1)(v^2)\vec{X}_1 \cdot \vec{X}_2 + (v^2)^2\vec{X}_2 \cdot \vec{X}_2 \\ &= g_{mn}v^m v^n (\text{recall } g_{mn} = \vec{X}_m \cdot \vec{X}_n, \text{ see page 35}). \end{aligned}$$

Therefore finding extrema of $k_n(\vec{v})$ for $\|\vec{v}\| = 1$ is equivalent to finding extrema of

$$k = k_n(\vec{v}) = \frac{L_{ij}v^i v^j}{g_{mn}v^m v^n}$$

for $\vec{v} \in T_{\vec{P}}(M)$ and $\vec{v} \neq \vec{0}$. If $k_n(\vec{v})$ is an extreme value of k, where $\vec{v} = v^i \vec{X}_i$, then $\dfrac{\partial k}{\partial v^1} = \dfrac{\partial k}{\partial v^2} = 0$ at $\vec{v}$ (that is, the gradient of k is $\vec{0}$; however, this gradient is computed in a (v^1, v^2) coordinate system, not (x, y)). Now

$$\frac{\partial k}{\partial v^r} = \frac{[2L_{rj}v^j](g_{mn}v^m v^n) - (L_{ij}v^i v^j)[2g_{rn}v^n]}{(g_{mn}v^m v^n)^2}$$

for $r = 1, 2$ (the derivatives in the numerator follow from Exercise 1.5.1). Now $k = \dfrac{L_{ij}v^i v^j}{g_{mn}v^m v^n}$, so replacing $L_{ij}v^i v^j$ with $kg_{mn}v^m v^n$ gives

$$\begin{aligned}
\frac{\partial k}{\partial v^r} &= \frac{2L_{rj}v^j(g_{mn}v^m v^n) - (kg_{mn}v^m v^n)2g_{rn}v^n}{(g_{mn}v^m v^n)^2} \\
&= \frac{2L_{rj}v^j - 2kg_{rn}v^n}{g_{mn}v^m v^n} = \frac{2L_{rj}v^j - 2kg_{rj}v^j}{g_{mn}v^m v^n} \\
&= \frac{2(L_{rj} - kg_{rj})v^j}{g_{mn}v^m v^n},
\end{aligned}$$

for $r = 1, 2$. So at an extreme value,

$$(L_{ij} - kg_{ij})v^j = 0 \text{ for } i = 1, 2. \qquad (24)$$

This is two linear equations in two unknowns (v^1 and v^2). Since $\vec{v}$ is nonzero, the only way this system can have a solution is for $\det(L_{ij} - kg_{ij}) = 0$. That is

$$\det \begin{bmatrix} L_{11} - kg_{11} & L_{12} - kg_{12} \\ L_{21} - kg_{21} & L_{22} - kg_{22} \end{bmatrix} = 0$$

$$\text{or} \quad (L_{11} - kg_{11})(L_{22} - kg_{22}) - (L_{21} - kg_{21})(L_{12} - kg_{12}) = 0$$

$$\text{or} \quad L_{11}L_{22} - kL_{11}g_{22} - kL_{22}g_{11} + k^2 g_{11}g_{22}$$

$$-L_{21}L_{12} + kL_{21}g_{12} + kL_{12}g_{21} - k^2 g_{12}g_{21} = 0$$

$$\text{or} \quad k^2(g_{11}g_{22} - g_{12}g_{21}) - k(g_{11}L_{22} + g_{22}L_{11}$$

$$-g_{12}L_{12} - g_{21}L_{21}) + (L_{11}L_{22} - L_{21}L_{12}) = 0$$

$$\text{or} \quad k^2 g - k(g_{11}L_{22} + g_{22}L_{11} - 2g_{12}L_{12}) + L = 0$$

since $L_{12} = L_{21}$, $L = \det(L_{ij})$, and $g = \det(g_{ij})$. So for extrema of k we need

$$k^2 - k\left(\frac{g_{11}L_{22} + g_{22}L_{11} - 2g_{12}L_{12}}{g}\right) + \frac{L}{g} = 0.$$

Since k_1 and k_2 are known to be roots of this equation, this equation factors as $(k - k_1)(k - k_2) = k^2 - (k_1 + k_2)k + k_1k_2 = 0$. Therefore, the Gauss curvature is $k_1k_2 = L/g$. ∎

Note. L is the determinant of the Second Fundamental form and g is the determinant of the First Fundamental Form. We now see good evidence for these being called "Fundamental" forms.

Example (Example 14, page 45 and Example 16, page 51). Consider the surface $\vec{X}(u, v) = (u, v, f(u, v))$. Then $\vec{X}_1 = (1, 0, f_u)$, $\vec{X}_2 = (0, 1, f_v)$, $\vec{X}_{11} = (0, 0, f_{uu})$, $\vec{X}_{22} = (0, 0, f_{vv})$, and $\vec{X}_{12} = \vec{X}_{21} = (0, 0, f_{uv})$. With $g_{ij} = \vec{X}_i \cdot \vec{X}_j$ we have

$$(g_{ij}) = \begin{pmatrix} 1 + f_u^2 & f_uf_v \\ f_uf_v & 1 + f_v^2 \end{pmatrix}$$

and so $g = \det(g_{ij}) = 1 + f_u^2 + f_v^2$. Now

$$\vec{X}_1 \times \vec{X}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)$$

and

$$\vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} = \frac{\vec{X}_1 \times \vec{X}_2}{\sqrt{g}} = \frac{1}{\sqrt{g}}(-f_u, -f_v, 1).$$

Next, $L_{ij} = \vec{X}_{ij} \cdot \vec{U}$, so

$$L_{11} = \frac{1}{\sqrt{g}} f_{uu} \qquad L_{12} = \frac{1}{\sqrt{g}} f_{uv}$$

$$L_{21} = \frac{1}{\sqrt{g}} f_{uv} \qquad L_{22} = \frac{1}{\sqrt{g}} f_{vv}.$$

Therefore $L = \det(L_{ij}) = \frac{1}{g}(f_{uu}f_{vv} - (f_{uv})^2)$. So the Gauss Curvature is

$$\frac{L}{g} = \frac{f_{uu}f_{vv} - (f_{uv})^2}{g^2} = \frac{f_{uu}f_{vv} - (f_{uv})^2}{(1 + f_u^2 + f_v^2)^2}.$$

Note. You may recall from Calculus 3 that a critical point of $z = f(x, y)$ was tested to see if it was a local maximum or minimum by considering $D = f_{xx}f_{yy} - (f_{xy})^2$ at the critical point. If $D < 0$, the surface has a saddle point. If $D > 0$ and $f_{xx} > 0$, it has a local minimum. If $D > 0$ and $f_{xx} < 0$, it has a local maximum. This all makes sense now in the light of curvature!

Theorem 1.6.A. If $\vec{v}$ and $\vec{w}$ are principal directions for surface M at point $\vec{P}$ corresponding to k_1 (maximum normal curvature at $\vec{P}$) and k_2 (minimum normal curvature at $\vec{P}$) respectively, then if $k_1 \neq k_2$ we have $\vec{v}$ and $\vec{w}$ orthogonal.

Proof. Let $\vec{v} = v^i \vec{X}_i$ and $\vec{w} = w^i \vec{X}_i$. As in Theorem I-5 (equation (24))

$$(L_{ij} - k_1 g_{ij})v^j = 0 \text{ for } i = 1, 2, \text{ and} \qquad (*)$$

$$(L_{ij} - k_2 g_{ij})w^j = 0 \text{ for } i = 1, 2. \qquad (**)$$

Equation $(*)$ is equivalent to

$$L_{ij}v^i = k_1 g_{ij} v^i \text{ for } j = 1, 2. \tag{25}$$

Equation $(**)$ implies

$$(L_{ij} - k_2 g_{ij})v^i w^j = 0$$

(we now sum over $i = 1, 2$). So

$$(L_{ij}v^i - k_2 g_{ij} v^i)w^j = 0$$

and from (25) we have

$$(k_1 g_{ij} v^i - k_2 g_{ij} v^i)w^j = 0$$

or

$$(k_1 - k_2) g_{ij} v^i w^j = 0.$$

Now $\vec{v} \cdot \vec{w} = g_{ij} v^i w^j$ (see page 35). Since $k_1 - k_2 \neq 0$, it must be that $\vec{v} \cdot \vec{w} = 0$. ∎

Note. We are now justified in referring to "two" principal directions. When we consider the Gauss curvature at a point, we deal with the normal curvature $k_n(\vec{v})$ at this point, where $\vec{v} = v^i \vec{X}_i$ (i takes on the values 1 and 2). So our collection of directions is a two dimensional space. Since we have shown (for $k_1 \neq k_2$) that the direction in which $k_n(\vec{v})$ equals k_1 and the direction in which $k_n(\vec{v})$ equals k_2 are orthogonal, there can be ONLY ONE direction in which $k_n(\vec{v})$ equals k_1 (well, ...plus or minus) and similarly for k_2. In the event that $k_1 = k_2$, we choose two directions $\vec{v}$ and $\vec{w}$ as principal directions where $\vec{v} \cdot \vec{w} = 0$.

Definition. Suppose $\vec{P} = \vec{X}(u_0^1, u_0^2)$ and let Ω be a neighborhood of (u_0^1, u_0^2) on which $\vec{X}$ is one-to-one with a continuous inverse $\vec{X}^{-1} : \vec{X}(\Omega) \to \Omega$. Define $\vec{U}(u^1, u^2)$ to be a unit normal vector to the surface M determined by $\vec{X}$ at point $\vec{X}(u^1, u^2)$ (recall that $\vec{U} = \vec{X}_1 \times \vec{X}_2 / \|\vec{X}_1 \times \vec{X}_2\|$). Therefore $\vec{U} : \vec{X}(\Omega) \to S^2$. $\vec{U}$ is called the *sphere mapping* or *Gauss mapping* of $\vec{X}(\Omega)$. The image of $\vec{X}(\Omega)$ under $\vec{U}$ (a subset of S^2) is the *spherical normal image* of $\vec{X}(\Omega)$.

Example (Exercise 9 (d), page 57). The spherical normal image of a torus (see Example 12, page 34) is the whole sphere S^2 (there is a normal vector pointing in any direction - in fact, the sphere mapping is two-to-one).

Lemma I-6. $\vec{U}_1 \times \vec{U}_2 = K(\vec{X}_1 \times \vec{X}_2)$.

Proof. Define

$$L_j^i = L_j^i(u^1, u^2) = L_{jk} g^{ki} \text{ for } i, j = 1, 2. \qquad (27)$$

Notice

$$L_j^i g_{im} = (L_{jk} g^{ki}) g_{im} = L_{jk} \delta_m^k = L_{jm} \qquad (27')$$

(recall (g^{ij}) is the inverse of (g_{ij})). Since $\vec{U} \cdot \vec{U} = 1$, $\vec{U} \cdot \vec{U}_j = 0$ (product rule) and so $\vec{U}_j$ is tangent to M. Therefore $\vec{U}_j$ is a linear combination of $\vec{X}_1$ and $\vec{X}_2$:

$$\vec{U}_j = a_j^r \vec{X}_r \text{ for } j = 1, 2$$

for some coefficients a_j^r. Since $\vec{U}$ is normal to M and $\vec{X}_k$ is tangent to M (at a given point) then $\vec{U} \cdot \vec{X}_k = 0$. Differentiating this equation with respect to u^j gives $\vec{U}_j \cdot \vec{X}_k + \vec{U} \cdot \vec{X}_{jk} = 0$ and so $\vec{U}_j \cdot \vec{X}_k = -\vec{U} \cdot \vec{X}_{jk} = -L_{jk}$ (this last equality follows from equation (2), page 44). So

$$-L_{jk} = \vec{U}_j \cdot \vec{X}_k = a_j^r \vec{X}_r \cdot \vec{X}_k = a_j^r g_{rk},$$

for $j, k = 1, 2$ (recall the definition of g_{rk}). We now solve these four equations ($j, k = 1, 2$) in the four unknowns a_j^r:

$$-L_{jk} = a_j^r g_{rk} \qquad (j, k = 1, 2)$$

$$-g^{ki} L_{jk} = a_j^r g_{rk} g^{ki} = a_j^r \delta_r^i = a_j^i \qquad (i, j = 1, 2).$$

Therefore (by the definition of L_j^i) $a_j^i = -L_j^i$. We now see how $\vec{U}_i$ and $\vec{X}_j$ relate:

$$\vec{U}_j = -L_j^i \vec{X}_i \text{ for } j = 1, 2.$$

From these relationships:

$$\begin{aligned}
\vec{U}_1 \times \vec{U}_2 &= (-L_1^i \vec{X}_i) \times (-L_2^k \vec{X}_k) \\
&= (-L_1^1 \vec{X}_1 - L_1^2 \vec{X}_2) \times (-L_2^1 \vec{X}_1 - L_2^2 \vec{X}_2) \\
&= (L_1^1 L_2^2 - L_1^2 L_2^1) \vec{X}_1 \times \vec{X}_2 \text{ (recall } \vec{v} \times \vec{v} = 0) \\
&= \det(L_j^i) \vec{X}_1 \times \vec{X}_2.
\end{aligned}$$

Since $L_j^i = L_{jk} g^{ki}$, then $\det(L_j^i) = \det(L_{jk}) \det(g^{ki})$ and since (g^{ki}) is the inverse of (g_{ki}),

$$\det(g^{ki}) = \frac{1}{\det(g_{ki})} = \frac{1}{g}$$

and so

$$\det(L_j^i) = \frac{\det(L_{jk})}{\det(g_{ki})} = \frac{L}{g} = K.$$

Therefore, $\vec{U}_1 \times \vec{U}_2 = K(\vec{X}_1 \times \vec{X}_2)$. ∎

Definition. For a surface determined by $\vec{X}(u^1, u^2)$, with $\vec{U}_j$, $\vec{X}_i$ and L_j^i defined as above, the equations $\vec{U}_j = -L_j^i \vec{X}_i$ for $j = 1, 2$ are the *equations of Weingarten.*

Note. For Ω a neigborhood of (u_0^1, u_0^2) on which $\vec{X}$ is one-to-one with a continuous inverse, the set $\vec{X}(\Omega)$ is a connected region on M. The spherical normal image of $\vec{X}(\Omega)$, $\vec{U}(\Omega)$ is a region on S^2 (see Figure I-26, page 52). If the curvature of $\vec{X}(\Omega)$ varies little then the area of $\vec{U}(\Omega)$ will be small. In fact, if $\vec{X}(\Omega)$ is part of a plane, then the area of $\vec{U}(\Omega)$ is zero. In fact, for Ω small, the ratio of the area of $\vec{U}(\Omega)$ to the area of $\vec{X}(\Omega)$ approximates the curvature of M on Ω.

Note. The tangent plane to S^2 at $\vec{U}(u^1, u^2)$, $T_{\vec{U}}S^2$, is parallel (that is, has the same normal vector) to the tangent plane to M at $\vec{X}(u^1, u^2)$, $T_{\vec{X}}M$. If $\vec{U}_1 \times \vec{U}_2 \neq \vec{0}$ (i.e. if $\vec{U}_1$ and $\vec{U}_2$ are linearly independent) then $\dfrac{\vec{U}_1 \times \vec{U}_2}{\|\vec{U}_1 \times \vec{U}_2\|}$ and $\vec{U}$ are both unit normal vectors to S^2 at the point $\vec{U}$ and do can differ at most in sign. That is, $\vec{U} = \pm \dfrac{\vec{U}_1 \times \vec{U}_2}{\|\vec{U}_1 \times \vec{U}_2\|}$ or $\vec{U}_1 \times \vec{U}_2 = \pm \vec{U}\|\vec{U}_1 \times \vec{U}_2\|$ or $\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = \pm \|\vec{U}_1 \times \vec{U}_2\|$ (recall $\vec{U} \cdot \vec{U} = 1$).

Note. If $(\vec{U}_1 \times \vec{U}_2)(u_0^1, u_0^2) \neq \vec{0}$ then $\vec{U}$ is regular at (u_0^1, u_0^2) (by definition) and therefore (by the comment on page 24) $\vec{U}$ is one-to-one with a continuous inverse on sufficiently small Ω, a neighborhood of (u_0^1, u_0^2). Also, with Ω sufficiently small, $\vec{U} \cdot \vec{U}_1 \times \vec{U}_2$ will be the same multiple of $\|\vec{U}_1 \times \vec{U}_2\|$ (namely $+1$ or -1). By equation (13), page 37,

$$\text{Area } U(\Omega) = \int\int_\Omega \|\vec{U}_1 \times \vec{U}_2\| du^1\, du^2$$

$$\text{Area } \vec{X}(\Omega) = \int\int_\Omega \|\vec{X}_1 \times \vec{X}_2\| du^1\, du^2.$$

Now

$$\vec{U} \cdot \vec{X}_1 \times \vec{X}_2 = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} \cdot (\vec{X}_1 \times \vec{X}_2) = \frac{\|\vec{X}_1 \times \vec{X}_2\|^2}{\|\vec{X}_1 \times \vec{X}_2\|} = \|\vec{X}_1 \times \vec{X}_2\|.$$

Also, we refer to $\displaystyle\int\!\!\int_\Omega \vec{U}\cdot\vec{U}_1\times\vec{U}_2\,du^1\,du^2$ as the *signed area* of $\vec{U}(\Omega)$ (recall it is $\pm$area of $\vec{U}(\Omega)$). Therefore

$$\text{signed area } \vec{U}(\Omega) = \int\!\!\int_\Omega \vec{U}\cdot\vec{U}_1\times\vec{U}_2\,du^1\,du^2$$

$$\text{area } \vec{X}(\Omega) = \int\!\!\int_\Omega \vec{U}\cdot\vec{X}_1\times\vec{X}_2\,du^1\,du^2.$$

Note. If $(\vec{U}_1\times\vec{U}_2)(u_0^1,u_0^2)=\vec{0}$, then notice that $\vec{U}\cdot\vec{U}_1\times\vec{U}_2$ may change sign and $\vec{U}$ may not be one-to-one over Ω and

$$\int\!\!\int_\Omega \vec{U}\cdot\vec{U}_1\times\vec{U}_2\,du^1\,du^2$$

then represents a "net area" of $\vec{U}(\Omega)$. In all these cases, we denote

$$\int\!\!\int_\Omega \vec{U}\cdot\vec{U}_1\times\vec{U}_2\,du^1\,du^2$$

as "Area $\vec{U}(\Omega)$" even though this is a bit of a misnomer.

Theorem 1.6.B. Suppose M is a surface determined by $\vec{X}(u^1,u^2)$ and $\vec{P} = \vec{X}(u_0^1,u_0^2)$ is a point on M. Let Ω be a neighborhood of (u_0^1,u_0^2) on which $\vec{X}$ is one-to-one with continuous inverse. Let $\vec{U}(\Omega)$ be the spherical normal image of $\vec{X}(\Omega)$. Then

$$K(P) = \lim_{\Omega\to(u_0^1,u_0^2)} \frac{\text{Area } \vec{U}(\Omega)}{\text{Area } \vec{X}(\Omega)}.$$

Here "Area $\vec{U}(\Omega)$" is as discussed above. The limit is taken in the sense that

$$\sup\{\ \text{dist}\ (\omega,(u_0^1,u_0^2)) \mid \omega\in\Omega\}$$

approaches zero.

Proof. Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that for Ω a ball with center (u_0^1, u_0^2) and radius δ_1 we have

$$\left| \text{Area } \vec{U}(\Omega) - (\vec{U} \cdot \vec{U}_1 \times \vec{U}_2)(\vec{P}) \text{ Area}(\Omega) \right|$$

$$= \left| \int\int_{\Omega} \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du^1 \, du^2 - (\vec{U} \cdot \vec{U}_1 \times \vec{U}_2)(\vec{P}) \text{ Area}(\Omega) \right| < \varepsilon$$

(since $\vec{U} \cdot \vec{U}_1 \times \vec{U}_2$ is continuous and Ω is connected). A similar result holds for Area $\vec{X}(\Omega)$. Therefore, for Ω sufficiently small,

$$\left| \frac{\text{Area } \vec{U}(\Omega)}{\text{Area } \vec{X}(\Omega)} - \frac{\vec{U} \cdot \vec{U}_1 \times \vec{U}_2}{\vec{U} \cdot \vec{X}_1 \times \vec{X}_2}(\vec{P}) \right| < \varepsilon.$$

That is,

$$\lim_{\Omega \to (u_0^1, u_0^2)} \frac{\text{Area } \vec{U}(\Omega)}{\text{Area } \vec{X}(\Omega)} = \frac{\vec{U} \cdot \vec{U}_1 \times \vec{U}_2}{\vec{U} \cdot \vec{X}_1 \times \vec{X}_2}.$$

By Lemma I-6,

$$\frac{\vec{U} \cdot \vec{U}_1 \times \vec{U}_2}{\vec{U} \cdot \vec{X}_1 \times \vec{X}_2} = \frac{\vec{U} \cdot K(\vec{X}_1 \times \vec{X}_2)}{\vec{U} \cdot \vec{X}_1 \times \vec{X}_2} = K$$

and the result follows. ∎

Example (Exercise 8 (a), page 56). Let $\vec{X} = \vec{X}(u, v)$ where $(u, v) \in D$ be a parameterization of a surface M. The (signed) area of the spherical normal image of M, $\int\int_D \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du \, dv$, is called the *total curvature* of M (assuming the integral, which may be improper, exists). Show that the total curvature of M is $\int\int_D K\sqrt{g} \, du \, dv$ (remember, K and g are functions of u and v).

Solution. By Lemma I-6, $\vec{U}_1 \times \vec{U}_2 = K(\vec{X}_1 \times \vec{X}_2)$. Therefore

$$\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = \vec{U} \cdot K(\vec{X}_1 \times \vec{X}_2).$$

Now the unit normal vector is $\vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|}$, so

$$\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = \frac{K(\vec{X}_1 \times \vec{X}_2) \cdot (\vec{X}_1 \times \vec{X}_2)}{\|\vec{X}_1 \times \vec{X}_2\|} = K\|\vec{X}_1 \times \vec{X}_2\|.$$

By equation (10), page 35, $\sqrt{g} = \|\vec{X}_1 \times \vec{X}_2\|$. Therefore

$$\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = K\sqrt{g}$$

and the total curvature of M over D is

$$\int\int_D \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du \, dv = \int\int_D K\sqrt{g} \, du \, dv.$$

∎

Example (Exercise 9 (d), page 57). Compute the total curvature of the torus

$$\vec{X}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).$$

Solution. From Example 12, page 34, and Exercise 1.4.3 (d), page 38,

$$\vec{U} = (-\cos u \cos v, -\cos u \sin v, -\sin u).$$

So

$$\begin{aligned} \vec{U}_1 &= \frac{\partial \vec{U}}{\partial u} = (\sin u \cos v, \sin u \sin v, -\cos u) \\ \vec{U}_2 &= \frac{\partial \vec{U}}{\partial v} = (\cos u \sin v, -\cos u \cos v, 0). \end{aligned}$$

Therefore

$$\begin{aligned} \vec{U}_1 \times \vec{U}_2 &= (-\cos^2 u \cos v, -\cos^2 u \sin v, -\sin u \cos u \cos^2 v \\ &\quad -\sin u \cos u \sin^2 v) \\ &= (-\cos^2 u \cos v, -\cos^2 u \sin v, -\sin u \cos u) \end{aligned}$$

and

$$\begin{aligned}\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 &= \cos^3 u \cos^2 v + \cos^3 u \sin^2 v + \sin^2 u \cos u \\ &= \cos^3 u + \sin^2 u \cos u.\end{aligned}$$

So the total curvature is

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\cos^3 u + \sin^2 u \cos u)\, du\, dv.$$

Now $\cos^3 u + \sin^2 u \cos u$ is an even function, so the integral is 0 and the total curvature is 0.

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