1.6 The Gauss Curvature in Detail

Note. We have defined the normal curvature of a surface at a point \( \vec{P} \) in the direction \( \vec{v} \): \( k_n(\vec{v}) \). Therefore, for a given point on a surface, there are an infinite number of (not necessarily distinct) curvatures (one for each “direction”). We can think of \( k_n(\vec{v}) \) as a function mapping the vector space \( T_{\vec{P}}(M) \) (the plane tangent to surface \( M \) at point \( \vec{P} \)) into \( \mathbb{R} \). That is \( k_n : T_{\vec{P}}(M) \rightarrow \mathbb{R} \). We need \( \vec{v} \) to be a unit vector, so the domain of \( k_n \) is \( \{ \vec{v} \in T_{\vec{P}}(M) \mid \|\vec{v}\| = 1 \} \). Therefore, \( k_n \) is a continuous function on a compact set and by the Extreme Value Theorem (for metric spaces), \( k_n \) assumes a maximum and a minimum value.

Definition. Let \( M \) be a surface and \( \vec{P} \) a point on the surface. Define \( k_1 = \max k_n(\vec{v}) \) and \( k_2 = \min k_n(\vec{v}) \) where the maximum and minimum are taken over the domain of \( k_n \). \( k_1 \) and \( k_2 \) are called the principal curvatures of \( M \) at \( \vec{P} \), and the corresponding directions are called principal directions. The product \( K = K(P) = k_1k_2 \) is the Gauss curvature of \( M \) at \( \vec{P} \).

Theorem I-5. The Gauss curvature at any point \( \vec{P} \) of a surface \( M \) is \( K(\vec{P}) = L/g \) where \( L = \det(L_{ij}) \) and \( g = \det(g_{ij}) \).

Proof. First, if \( \vec{v} = v^i\vec{X}_i \) then
\[
\|\vec{v}\|^2 = (v^1\vec{X}_1 + v^2\vec{X}_2) \cdot (v^1\vec{X}_1 + v^2\vec{X}_2) \\
= (v^1)^2\vec{X}_1 \cdot \vec{X}_1 + 2(v^1)(v^2)\vec{X}_1 \cdot \vec{X}_2 + (v^2)^2\vec{X}_2 \cdot \vec{X}_2 \\
= g_{mn}v^mv^n \text{ (recall } g_{mn} = \vec{X}_m \cdot \vec{X}_n, \text{ see page 35).}
\]
Therefore finding extrema of $k_n(\vec{v})$ for $\|\vec{v}\| = 1$ is equivalent to finding extrema of

$$k = k_n(\vec{v}) = \frac{L_{ij}v^iv^j}{g_{mn}v^mv^n}$$

for $\vec{v} \in T_{\vec{p}}(M)$ and $\vec{v} \neq \vec{0}$. If $k_n(\vec{v})$ is an extreme value of $k$, where $\vec{v} = v^i\vec{X}_i$, then $\frac{\partial k}{\partial v^1} = \frac{\partial k}{\partial v^2} = 0$ at $\vec{v}$ (that is, the gradient of $k$ is $\vec{0}$; however, this gradient is computed in a $(v^1, v^2)$ coordinate system, not $(x, y)$). Now

$$\frac{\partial k}{\partial v^r} = \frac{[2L_{rj}v^j](g_{mn}v^mv^n) - (L_{ij}v^iv^j)[2g_{rn}v^n]}{(g_{mn}v^mv^n)^2}$$

for $r = 1, 2$ (the derivatives in the numerator follow from Exercise 1.5.1). Now $k = \frac{L_{ij}v^iv^j}{g_{mn}v^mv^n}$, so replacing $L_{ij}v^iv^j$ with $kg_{mn}v^mv^n$ gives

$$\frac{\partial k}{\partial v^r} = \frac{2L_{rj}v^j(g_{mn}v^mv^n) - (kg_{mn}v^mv^n)2g_{rn}v^n}{(g_{mn}v^mv^n)^2} = \frac{2L_{rj}v^j - 2kg_{rn}v^n}{g_{mn}v^mv^n} = \frac{2(L_{rj} - kg_{rj})v^j}{g_{mn}v^mv^n},$$

for $r = 1, 2$. So at an extreme value,

$$(L_{ij} - kg_{ij})v^j = 0 \text{ for } i = 1, 2. \quad (24)$$

This is two linear equations in two unknowns $(v^1$ and $v^2$). Since $\vec{v}$ is nonzero, the only way this system can have a solution is for det$(L_{ij} - kg_{ij}) = 0$. That is

$$\det \begin{bmatrix} L_{11} - kg_{11} & L_{12} - kg_{12} \\ L_{21} - kg_{21} & L_{22} - kg_{22} \end{bmatrix} = 0$$

or $(L_{11} - kg_{11})(L_{22} - kg_{22}) - (L_{21} - kg_{21})(L_{12} - kg_{12}) = 0$

or $L_{11}L_{22} - kL_{11}g_{22} - kL_{22}g_{11} + k^2g_{11}g_{22}$
\[-L_{21}L_{12} + kL_{21}g_{12} + kL_{12}g_{21} - k^2g_{12}g_{21} = 0\]

or
\[k^2(g_{11}g_{22} - g_{12}g_{21}) - k(g_{11}L_{22} + g_{22}L_{11} - g_{12}L_{12}) = 0\]

or
\[k^2g - k(g_{11}L_{22} + g_{22}L_{11} - 2g_{12}L_{12}) + L = 0\]

since \(L_{12} = L_{21}\), \(L = \det(L_{ij})\), and \(g = \det(g_{ij})\). So for extrema of \(k\) we need
\[k^2 - k \left( \frac{g_{11}L_{22} + g_{22}L_{11} - 2g_{12}L_{12}}{g} \right) + \frac{L}{g} = 0.\]

Since \(k_1\) and \(k_2\) are known to be roots of this equation, this equation factors as
\[(k - k_1)(k - k_2) = k^2 - (k_1 + k_2)k + k_1k_2 = 0.\]

Therefore, the Gauss curvature is \(k_1k_2 = L/g\). \(\blacksquare\)

**Note.** \(L\) is the determinant of the Second Fundamental form and \(g\) is the determinant of the First Fundamental Form. We now see good evidence for these being called “Fundamental” forms.

**Example (Example 14, page 45 and Example 16, page 51).** Consider the surface \(\vec{X}(u, v) = (u, v, f(u, v))\). Then \(\vec{X}_1 = (1, 0, f_u), \vec{X}_2 = (0, 1, f_v), \vec{X}_{11} = (0, 0, f_{uu}), \vec{X}_{22} = (0, 0, f_{vv}),\) and \(\vec{X}_{12} = \vec{X}_{21} = (0, 0, f_{uv})\). With \(g_{ij} = \vec{X}_i \cdot \vec{X}_j\) we have
\[
(g_{ij}) = \begin{pmatrix}
1 + f_u^2 & f_u f_v \\
f_u f_v & 1 + f_v^2
\end{pmatrix}
\]

and so \(g = \det(g_{ij}) = 1 + f_u^2 + f_v^2\). Now
\[
\vec{X}_1 \times \vec{X}_2 = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 0 & f_u \\
0 & 1 & f_v
\end{vmatrix} = (-f_u, -f_v, 1)
and
\[ \mathbf{U} = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{\|\mathbf{X}_1 \times \mathbf{X}_2\|} = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{\sqrt{g}} = \frac{1}{\sqrt{g}}(-f_u, -f_v, 1). \]

Next, \( L_{ij} = \mathbf{X}_{ij} \cdot \mathbf{U} \), so
\[ L_{11} = \frac{1}{\sqrt{g}} f_{uu} \quad L_{12} = \frac{1}{\sqrt{g}} f_{uv}, \]
\[ L_{21} = \frac{1}{\sqrt{g}} f_{uv} \quad L_{22} = \frac{1}{\sqrt{g}} f_{vv}. \]

Therefore \( L = \det(L_{ij}) = \frac{1}{g} (f_{uu} f_{vv} - (f_{uv})^2) \). So the Gauss Curvature is
\[ \frac{L}{g} = \frac{f_{uu} f_{vv} - (f_{uv})^2}{g^2} = \frac{f_{uu} f_{vv} - (f_{uv})^2}{(1 + f_u^2 + f_v^2)^2}. \]

Note. You may recall from Calculus 3 that a critical point of \( z = f(x, y) \) was tested to see if it was a local maximum or minimum by considering \( D = f_{xx} f_{yy} - (f_{xy})^2 \) at the critical point. If \( D < 0 \), the surface has a saddle point. If \( D > 0 \) and \( f_{xx} > 0 \), it has a local minimum. If \( D > 0 \) and \( f_{xx} < 0 \), it has a local maximum. This all makes sense now in the light of curvature!

**Theorem 1.6.A.** If \( \mathbf{v} \) and \( \mathbf{w} \) are principal directions for surface \( M \) at point \( \mathbf{P} \) corresponding to \( k_1 \) (maximum normal curvature at \( \mathbf{P} \)) and \( k_2 \) (minimum normal curvature at \( \mathbf{P} \)) respectively, then if \( k_1 \neq k_2 \) we have \( \mathbf{v} \) and \( \mathbf{w} \) orthogonal.

**Proof.** Let \( \mathbf{v} = v^i \mathbf{X}_i \) and \( \mathbf{w} = w^i \mathbf{X}_i \). As in Theorem I-5 (equation (24))
\[ (L_{ij} - k_1 g_{ij}) v^j = 0 \quad \text{for} \ i = 1, 2, \ \text{and} \quad (*) \]
\[ (L_{ij} - k_2 g_{ij}) w^j = 0 \quad \text{for} \ i = 1, 2. \quad (**) \]
Equation (*) is equivalent to

\[ L_{ij} v^i = k_1 g_{ij} v^i \text{ for } j = 1, 2. \]  
(25)

Equation (**) implies

\[ (L_{ij} - k_2 g_{ij}) v^i w^j = 0 \]

(we now sum over \( i = 1, 2 \)). So

\[ (L_{ij} v^i - k_2 g_{ij} v^i) w^j = 0 \]

and from (25) we have

\[ (k_1 g_{ij} v^i - k_2 g_{ij} v^i) w^j = 0 \]

or

\[ (k_1 - k_2) g_{ij} v^i w^j = 0. \]

Now \( \vec{v} \cdot \vec{w} = g_{ij} v^i w^j \) (see page 35). Since \( k_1 - k_2 \neq 0 \), it must be that \( \vec{v} \cdot \vec{w} = 0. \)

**Note.** We are now justified in referring to “two” principal directions. When we consider the Gauss curvature at a point, we deal with the normal curvature \( k_n(\vec{v}) \) at this point, where \( \vec{v} = v^i \vec{X}_i \) (\( i \) takes on the values 1 and 2). So our collection of directions is a two dimensional space. Since we have shown (for \( k_1 \neq k_2 \)) that the direction in which \( k_n(\vec{v}) \) equals \( k_1 \) and the direction in which \( k_n(\vec{v}) \) equals \( k_2 \) are orthogonal, there can be ONLY ONE direction in which \( k_n(\vec{v}) \) equals \( k_1 \) (well, \( \ldots \) plus or minus) and similarly for \( k_2 \). In the event that \( k_1 = k_2 \), we choose two directions \( \vec{v} \) and \( \vec{w} \) as principal directions where \( \vec{v} \cdot \vec{w} = 0. \)
Definition. Suppose $\vec{P} = \vec{X}(u_0, u_0^2)$ and let $\Omega$ be a neighborhood of $(u_0^1, u_0^2)$ on which $\vec{X}$ is one-to-one with a continuous inverse $\vec{X}^{-1} : \vec{X}(\Omega) \to \Omega$. Define $\vec{U}(u^1, u^2)$ to be a unit normal vector to the surface $M$ determined by $\vec{X}$ at point $\vec{X}(u^1, u^2)$ (recall that $\vec{U} = \vec{X}_1 \times \vec{X}_2/\|\vec{X}_1 \times \vec{X}_2\|$). Therefore $\vec{U} : \vec{X}(\Omega) \to S^2$. $\vec{U}$ is called the sphere mapping or Gauss mapping of $\vec{X}(\Omega)$. The image of $\vec{X}(\Omega)$ under $\vec{U}$ (a subset of $S^2$) is the spherical normal image of $\vec{X}(\Omega)$.

Example (Exercise 9 (d), page 57). The spherical normal image of a torus (see Example 12, page 34) is the whole sphere $S^2$ (there is a normal vector pointing in any direction - in fact, the sphere mapping is two-to-one).

Lemma I-6. $\vec{U}_1 \times \vec{U}_2 = K(\vec{X}_1 \times \vec{X}_2)$.

Proof. Define

$$L^i_j = L^i_j(u^1, u^2) = L^i_{jk}g^{ki} \text{ for } i, j = 1, 2. \quad (27)$$

Notice

$$L^i_j g_{im} = (L^i_{jk}g^{ki})g_{im} = L^i_{jk} \delta^k_m = L^i_{jm} \quad (27')$$

(recall $(g^{ij})$ is the inverse of $(g_{ij})$). Since $\vec{U} \cdot \vec{U} = 1$, $\vec{U} \cdot \vec{U}_j = 0$ (product rule) and so $\vec{U}_j$ is tangent to $M$. Therefore $\vec{U}_j$ is a linear combination of $\vec{X}_1$ and $\vec{X}_2$:

$$\vec{U}_j = a^r_j \vec{X}_r \text{ for } j = 1, 2$$

for some coefficients $a^r_j$. Since $\vec{U}$ is normal to $M$ and $\vec{X}_k$ is tangent to $M$ (at a given point) then $\vec{U} \cdot \vec{X}_k = 0$. Differentiating this equation with respect to $u^j$ gives $\vec{U}_j \cdot \vec{X}_k + \vec{U} \cdot \vec{X}_{jk} = 0$ and so $\vec{U}_j \cdot \vec{X}_k = -\vec{U} \cdot \vec{X}_{jk} = -L^i_{jk}$ (this last equality follows
from equation (2), page 44). So

\[ -L_{jk} = \vec{U}_j \cdot \vec{X}_k = a^r_j \vec{X}_r \cdot \vec{X}_k = a^r_j g_{rk}, \]

for \( j, k = 1, 2 \) (recall the definition of \( g_{rk} \)). We now solve these four equations \((j, k = 1, 2)\) in the four unknowns \( a^r_j \):

\[ -L_{jk} = a^r_j g_{rk} \quad (j, k = 1, 2) \]

\[ -g^{ki} L_{jk} = a^r_j g_{rk} g^{ki} = a^r_j \delta_r^i = a^i_j \quad (i, j = 1, 2). \]

Therefore (by the definition of \( L^i_j \)) \( a^i_j = -L^i_j \). We now see how \( \vec{U}_i \) and \( \vec{X}_j \) relate:

\[ \vec{U}_j = -L^i_j \vec{X}_i \quad \text{for} \quad j = 1, 2. \]

From these relationships:

\[ \vec{U}_1 \times \vec{U}_2 = (-L^i_1 \vec{X}_i) \times (-L^k_2 \vec{X}_k) = (-L^1_1 \vec{X}_1 - L^2_1 \vec{X}_2) \times (-L^1_2 \vec{X}_1 - L^2_2 \vec{X}_2) = (L^1_1 L^2_2 - L^1_2 L^2_1) \vec{X}_1 \times \vec{X}_2 \quad (\text{recall} \quad \vec{v} \times \vec{v} = 0) \]

\[ = \det(L^i_j) \vec{X}_1 \times \vec{X}_2. \]

Since \( L^i_j = L_{jk} g^{ki} \), then \( \det(L^i_j) = \det(L_{jk}) \det(g^{ki}) \) and since \( (g^{ki}) \) is the inverse of \( (g_{ki}) \),

\[ \det(g^{ki}) = \frac{1}{\det(g_{ki})} = \frac{1}{g} \]

and so

\[ \det(L^i_j) = \frac{\det(L_{jk})}{\det(g_{ki})} = \frac{L}{g} = K. \]

Therefore, \( \vec{U}_1 \times \vec{U}_2 = K(\vec{X}_1 \times \vec{X}_2) \).

Definition. For a surface determined by \( \vec{X}(u^1, u^2) \), with \( \vec{U}_j, \vec{X}_i \) and \( L^i_j \) defined as above, the equations \( \vec{U}_j = -L^i_j \vec{X}_i \) for \( j = 1, 2 \) are the \textit{equations of Weingarten}.\[\]
Note. For Ω a neighborhood of \((u_0^1, u_0^2)\) on which \(\vec{X}\) is one-to-one with a continuous inverse, the set \(\vec{X}(\Omega)\) is a connected region on \(M\). The spherical normal image of \(\vec{X}(\Omega), \vec{U}(\Omega)\) is a region on \(S^2\) (see Figure I-26, page 52). If the curvature of \(\vec{X}(\Omega)\) varies little then the area of \(\vec{U}(\Omega)\) will be small. In fact, if \(\vec{X}(\Omega)\) is part of a plane, then the area of \(\vec{U}(\Omega)\) is zero. In fact, for \(\Omega\) small, the ratio of the area of \(\vec{U}(\Omega)\) to the area of \(\vec{X}(\Omega)\) approximates the curvature of \(M\) on \(\Omega\).

Note. The tangent plane to \(S^2\) at \(\vec{U}(u^1, u^2), T_{\vec{U}}S^2\), is parallel (that is, has the same normal vector) to the tangent plane to \(M\) at \(\vec{X}(u^1, u^2), T_{\vec{X}}M\). If \(\vec{U}_1 \times \vec{U}_2 \neq \vec{0}\) (i.e. if \(\vec{U}_1\) and \(\vec{U}_2\) are linearly independent) then \(\frac{\vec{U}_1 \times \vec{U}_2}{\|\vec{U}_1 \times \vec{U}_2\|}\) and \(\vec{U}\) are both unit normal vectors to \(S^2\) at the point \(\vec{U}\) and do can differ at most in sign. That is, \(\vec{U} = \pm \frac{\vec{U}_1 \times \vec{U}_2}{\|\vec{U}_1 \times \vec{U}_2\|}\) or \(\vec{U}_1 \times \vec{U}_2 = \pm \vec{U}\|\vec{U}_1 \times \vec{U}_2\|\) or \(\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = \pm \|\vec{U}_1 \times \vec{U}_2\|\) (recall \(\vec{U} \cdot \vec{U} = 1\)).

Note. If \((\vec{U}_1 \times \vec{U}_2)(u_0^1, u_0^2) \neq \vec{0}\) then \(\vec{U}\) is regular at \((u_0^1, u_0^2)\) (by definition) and therefore (by the comment on page 24) \(\vec{U}\) is one-to-one with a continuous inverse on sufficiently small \(\Omega\), a neighborhood of \((u_0^1, u_0^2)\). Also, with \(\Omega\) sufficiently small, \(\vec{U} \cdot \vec{U}_1 \times \vec{U}_2\) will be the same multiple of \(\|\vec{U}_1 \times \vec{U}_2\|\) (namely +1 or −1). By equation (13), page 37,

\[
\text{Area } U(\Omega) = \int \int_\Omega \|\vec{U}_1 \times \vec{U}_2\| du^1 du^2
\]

\[
\text{Area } \vec{X}(\Omega) = \int \int_\Omega \|\vec{X}_1 \times \vec{X}_2\| du^1 du^2.
\]

Now

\[
\vec{U} \cdot \vec{X}_1 \times \vec{X}_2 = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} \cdot (\vec{X}_1 \times \vec{X}_2) = \frac{\|\vec{X}_1 \times \vec{X}_2\|^2}{\|\vec{X}_1 \times \vec{X}_2\|} = \|\vec{X}_1 \times \vec{X}_2\|.
\]
Also, we refer to $\int \int_{\Omega} \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du^1 \, du^2$ as the signed area of $\vec{U}(\Omega)$ (recall it is $\pm \text{area of } \vec{U}(\Omega)$). Therefore

$$\text{signed area } \vec{U}(\Omega) = \int \int_{\Omega} \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du^1 \, du^2$$

$$\text{area } \vec{X}(\Omega) = \int \int_{\Omega} \vec{U} \cdot \vec{X}_1 \times \vec{X}_2 \, du^1 \, du^2.$$ 

Note. If $(\vec{U}_1 \times \vec{U}_2)(u^1_0, u^2_0) = \vec{0}$, then notice that $\vec{U} \cdot \vec{U}_1 \times \vec{U}_2$ may change sign and $\vec{U}$ may not be one-to-one over $\Omega$ and

$$\int \int_{\Omega} \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du^1 \, du^2$$

then represents a “net area” of $\vec{U}(\Omega)$. In all these cases, we denote

$$\int \int_{\Omega} \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du^1 \, du^2$$

as “Area $\vec{U}(\Omega)$” even though this is a bit of a misnomer.

**Theorem 1.6.B.** Suppose $M$ is a surface determined by $\vec{X}(u^1, u^2)$ and $\vec{P} = \vec{X}(u^1_0, u^2_0)$ is a point on $M$. Let $\Omega$ be a neighborhood of $(u^1_0, u^2_0)$ on which $\vec{X}$ is one-to-one with continuous inverse. Let $\vec{U}(\Omega)$ be the spherical normal image of $\vec{X}(\Omega)$. Then

$$K(P) = \lim_{\Omega \to (u^1_0, u^2_0)} \frac{\text{Area } \vec{U}(\Omega)}{\text{Area } \vec{X}(\Omega)}.$$ 

Here “Area $\vec{U}(\Omega)$” is as discussed above. The limit is taken in the sense that

$$\sup \{ \text{dist } (\omega, (u^1_0, u^2_0)) \mid \omega \in \Omega \}$$
approaches zero.

**Proof.** Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that for $\Omega$ a ball with center $(u_0^1, u_0^2)$ and radius $\delta_1$ we have

$$\left| \text{Area } \tilde{U}(\Omega) - (\tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2)(\tilde{P}) \text{ Area}(\Omega) \right| = \left| \int \int_{\Omega} \tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2 \, du^1 \, du^2 - (\tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2)(\tilde{P}) \text{ Area}(\Omega) \right| < \varepsilon \quad (\text{since } \tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2 \text{ is continuous and } \Omega \text{ is connected}).$$

A similar result holds for Area $\tilde{X}(\Omega)$. Therefore, for $\Omega$ sufficiently small,

$$\left| \frac{\text{Area } \tilde{U}(\Omega)}{\text{Area } \tilde{X}(\Omega)} - \frac{\tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2}{\tilde{U} \cdot \tilde{X}_1 \times \tilde{X}_2}(\tilde{P}) \right| < \varepsilon.$$

That is,

$$\lim_{\Omega \to (u_0^1, u_0^2)} \frac{\text{Area } \tilde{U}(\Omega)}{\text{Area } \tilde{X}(\Omega)} = \frac{\tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2}{\tilde{U} \cdot \tilde{X}_1 \times \tilde{X}_2}.$$

By Lemma I-6,

$$\frac{\tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2}{\tilde{U} \cdot \tilde{X}_1 \times \tilde{X}_2} = \frac{\tilde{U} \cdot K(\tilde{X}_1 \times \tilde{X}_2)}{\tilde{U} \cdot \tilde{X}_1 \times \tilde{X}_2} = K$$

and the result follows.

---

**Example (Exercise 8 (a), page 56).** Let $\tilde{X} = \tilde{X}(u, v)$ where $(u, v) \in D$ be a parameterization of a surface $M$. The (signed) area of the spherical normal image of $M$, $\int \int_D \tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2 \, du \, dv$, is called the *total curvature* of $M$ (assuming the integral, which may be improper, exists). Show that the total curvature of $M$ is $\int \int_D K \sqrt{g} \, du \, dv$ (remember, $K$ and $g$ are functions of $u$ and $v$).

**Solution.** By Lemma I-6, $\tilde{U}_1 \times \tilde{U}_2 = K(\tilde{X}_1 \times \tilde{X}_2)$. Therefore

$$\tilde{U} \cdot \tilde{U}_1 \times \tilde{U}_2 = \tilde{U} \cdot K(\tilde{X}_1 \times \tilde{X}_2).$$
Now the unit normal vector is \( \vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} \), so

\[
\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = \frac{K(\vec{X}_1 \times \vec{X}_2) \cdot (\vec{X}_1 \times \vec{X}_2)}{\|\vec{X}_1 \times \vec{X}_2\|} = K \|\vec{X}_1 \times \vec{X}_2\|.
\]

By equation (10), page 35, \( \sqrt{g} = \|\vec{X}_1 \times \vec{X}_2\| \). Therefore

\[
\vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = K \sqrt{g}
\]

and the total curvature of \( M \) over \( D \) is

\[
\int \int_D \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 \, du \, dv = \int \int_D K \sqrt{g} \, du \, dv.
\]

\[\Box\]

**Example (Exercise 9 (d), page 57).** Compute the total curvature of the torus

\[
\vec{X}(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u).
\]

**Solution.** From Example 12, page 34, and Exercise 1.4.3 (d), page 38,

\(
\vec{U} = (- \cos u \cos v, - \cos u \sin v, - \sin u).
\)

So

\[
\vec{U}_1 = \frac{\partial \vec{U}}{\partial u} = (\sin u \cos v, \sin u \sin v, - \cos u),
\]

\[
\vec{U}_2 = \frac{\partial \vec{U}}{\partial v} = (\cos u \sin v, - \cos u \cos v, 0).
\]

Therefore

\[
\vec{U}_1 \times \vec{U}_2 = (- \cos^2 u \cos v, - \cos^2 u \sin v, - \sin u \cos u \cos^2 v
\]

\[
- \sin u \cos u \sin^2 v)
\]

\[
= (- \cos^2 u \cos v, - \cos^2 u \sin v, - \sin u \cos u)
\]
and

\[ \vec{U} \cdot \vec{U}_1 \times \vec{U}_2 = \cos^3 u \cos^2 v + \cos^3 u \sin^2 v + \sin^2 u \cos u \]

\[ = \cos^3 u + \sin^2 u \cos u. \]

So the total curvature is

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\cos^3 u + \sin^2 u \cos u) \, du \, dv. \]

Now \( \cos^3 u + \sin^2 u \cos u \) is an even function, so the integral is 0 and the total curvature is 0.

Revised: 6/13/2016