1.9 Manifolds

**Note.** In this section, we extend the ideas of tangents, metrics, geodesics, and curvature to “manifolds” (in a sense, “$n$–dimensional surfaces”) without appealing to how they are embedded in a higher dimensional space.

**Definition.** Let $M$ be a non-empty set whose elements we call points. A *coordinate patch* on $M$ is a one-to-one function $\vec{X} : D \to M$ (continuous and regular) from an open subset $D$ of $E^2$ (or more generally $E^n$) into $M$.

**Note.** In the following definition, by “domain” of a function we mean the largest set on which the function is defined. By “smooth” we mean sufficiently differentiable for our purposes. A function whose domain is empty is considered smooth.

**Definition I-12.** An *abstract surface* or *2–manifold* (more generally, *$n$–manifold*) is a set $M$ with a collection $\mathcal{C}$ of coordinate patches on $M$ satisfying:

(a) $M$ is the union of images of the patches in $\mathcal{C}$ (that is, if $\mathcal{C} = \{\vec{X}^i\}$ and $\vec{X}^i$ is defined on set $D^i$, then $M = \bigcup_i \vec{X}^i(D^i)$).

(b) The patches of $\mathcal{C}$ *overlap smoothly*, that is if $\vec{X}^1 : D^1 \to M$ and $\vec{X}^2 : D^2 \to M$ are two patches in $\mathcal{C}$, then $(\vec{X}^1)^{-1} \circ \vec{X}^2$ and $(\vec{X}^2)^{-1} \circ \vec{X}^1$ have open domains and are smooth.

(c) Given two points $\vec{P}^1$ and $\vec{P}^2$ of $M$, there exist coordinate patches $\vec{X}^1 : D^1 \to M$ and $\vec{X}^2 : D^2 \to M$ in $\mathcal{C}$ such that $\vec{P}^1 \in \vec{X}^1(D^1)$, $\vec{P}^2 \in \vec{X}^2(D^2)$ and
\[ \bar{X}^1(D^1) \cap \bar{X}^2(D^2) = \emptyset \] (this is the Hausdorff property).

(d) The collection \( \mathcal{C} \) is maximal. That is, any coordinate patch on \( M \) which overlaps smoothly with every patch of \( \mathcal{C} \) is itself in \( \mathcal{C} \). (Notice that two disjoint coordinate patches “overlap smoothly” by convention).

**Definition.** The collection \( \mathcal{C} \) is called a differentiable structure on \( M \) and patches in \( \mathcal{C} \) are called admissible patches.

**Note.** If properties (a), (b), and (c) of Definition I-12 are satisfied by a collection \( \mathcal{C}' \) then we can adjoin to \( \mathcal{C}' \) all patches that overlap smoothly with the patches of \( \mathcal{C}' \) to create a collection \( \mathcal{C} \) which satisfies (a), (b), (c), (d). In this case, \( \mathcal{C}' \) is said to generate \( \mathcal{C} \).
Example (Exercise 1.9.1). Let $M$ be the plane with Cartesian coordinates. The identity mapping of $M$ onto itself is a coordinate patch. A differentiable structure on $M$ is obtained by adjoining to this mapping all patches in $M$ which overlap smoothly with this mapping. The polar coordinate patch

$$u = r \cos \theta \quad v = r \sin \theta \quad (r, \theta) \in D$$

overlaps smoothly with the identity patch IF $D$ is of the form

$$D = \{(r, \theta) \mid r > 0, \theta \in (a, b), b - a \leq 2\pi\}$$

(an open sector).

Solution. Let $\vec{X} : D \to M$ and $\vec{X} : \overline{D} \to M$. By definition, $\vec{X}$ and $\vec{X}$ overlap smoothly if $(\vec{X})^{-1} \circ \vec{X}$ (which maps $\overline{D} \to M \to D$) and $(\vec{X})^{-1} \circ \vec{X}$ (which maps $D \to M \to \overline{D}$) have open domains and are smooth.

First, $\vec{X}$ and $\vec{X}$ are one-to-one and so are invertible. Explicitly,

$$(\vec{X})^{-1} \circ \vec{X}(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta).$$

So

$$\frac{\partial}{\partial r}[(\vec{X})^{-1} \circ \vec{X}] = (\cos \theta, \sin \theta)$$

and

$$\frac{\partial}{\partial \theta}[(\vec{X})^{-1} \circ \vec{X}] = (-r \sin \theta, r \cos \theta).$$

Therefore, $(\vec{X})^{-1} \circ \vec{X}$ is smooth (the first partials are continuous... in fact, it is infinitely differentiable). Similarly, $(\vec{X})^{-1} \circ \vec{X}(x, y) = (r, \theta)$ where $r = \sqrt{x^2 + y^2}$ and $\tan \theta = y/x$ where we choose $\theta$ such that $\theta \in (a, b)$,

- $\theta$ is in Quadrant I if $x > 0$, $y > 0$,
- $\theta$ is in Quadrant II if $x < 0$, $y > 0$,
\( \theta \) is in Quadrant III if \( x < 0, y < 0 \),
\( \theta \) is in Quadrant IV if \( x > 0, y < 0 \),
(and similar choices are made if \( x = 0 \) or \( y = 0 \)). So \( \theta = \tan^{-1}(y/x) + \text{constant}_\theta \)
so \( \theta \) is a continuous function of \((x, y)\), even though \( \tan^{-1}(y/x) \) is not continuous
— this is how we choose the \( \theta \) to associate with \((x, y)\)). We then have
\[
\frac{\partial}{\partial x}[(\vec{X}^{-1} \circ \vec{X})] = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{-y/x^2}{1 + (y/x)^2} \right)
\]
and
\[
\frac{\partial}{\partial y}[(\vec{X}^{-1} \circ \vec{X})] = \left( \frac{y}{\sqrt{x^2 + y^2}}, \frac{1/x}{1 + (y/x)^2} \right),
\]
therefore (since \((x, y) \neq (0, 0)\)) \( (\vec{X})^{-1} \circ \vec{X} \) is smooth (in fact, infinitely differen-
tiable). Next, the domain of \( (\vec{X})^{-1} \circ \vec{X} : D \to \overline{D} \) is \( D \) itself and \( D \) is open (by
definition). The domain of \( (\vec{X})^{-1} \circ \vec{X} : \overline{D} \to D \) is the set of all \((x, y) \in M \) such
that \( \sqrt{x^2 + y^2} = r > 0 \) and \( \tan^{-1}(y/x) \in (a, b) \) (where \( \tan^{-1}(y/x) \) is calculated as
described above). Therefore the domain of \( (\vec{X})^{-1} \circ \vec{X} \) is open. Hence, \( \vec{X} \) and \( \vec{X} \)
overlap smoothly.

**Definition.** An admissible patch \( \vec{X} : D \to M \) associates with each point \( \vec{P} \) of
\( \vec{X}(D) \) a unique ordered pair (or in general, ordered \( n \)-tuple) \((u^1, u^2) = \vec{X}^{-1}(\vec{P}) \)
called a local coordinate of \( \vec{P} \) with respect to \( \vec{X} \).

**Note.** A point \( \vec{P} \) can have different local coordinates with respect to different
admissible patches. Suppose, for example, \( \vec{P} = \vec{X}(u^1, u^2) = \vec{X}(\vec{w}^1, \vec{w}^2) \). Then
\( (\vec{X})^{-1} \circ \vec{X}(\vec{w}^1, \vec{w}^2) = (u^1, u^2) \) and \( (\vec{X})^{-1} \circ \vec{X}^1(u^1, u^2) = (\vec{w}^1, \vec{w}^2) \).
Definition. In the above setting, the equations \((\vec{X})^{-1} \circ \vec{X}(\overrightarrow{u^1}, \overrightarrow{u^2}) = \overrightarrow{u^1} \) and \((\vec{X})^{-1} \circ \vec{X}(u^1, u^1) = (\overrightarrow{u^1}, \overrightarrow{u^2})\) are changes of coordinates. See Figure I-29, page 83 (a form of which is above, after the definition of 2-manifold). In terms of local coordinates:

\[
\begin{align*}
\vec{X}^{-1} \circ \vec{X} & \text{ is given by } \overrightarrow{u^i} = \overrightarrow{u^i}(\overrightarrow{u^1}, \overrightarrow{u^2}), \ i = 1, 2 \quad (61a) \\
\vec{X}^{-1} \circ \vec{X} & \text{ is given by } u^i = u^i(\overrightarrow{u^1}, \overrightarrow{u^2}), \ i = 1, 2. \quad (61b)
\end{align*}
\]

Definition I-13. A set \(\Omega \subset M\) is a neighborhood of a point \(\vec{P} \in M\) if there exists an admissible patch \(\vec{X} : D \rightarrow M\) such that \(\vec{P} \in \vec{X}(D)\) and \(\vec{X}(D) \subset \Omega\). A subset of \(M\) is open if it is a neighborhood of each of its points.

Definition. Let \(\Omega\) be an open subset of the 2-manifold (or generally \(n\)-manifold) \(M\). A function \(f : \Omega \rightarrow \mathbb{R}\) is smooth if \(f \circ \vec{X}\) is smooth for every admissible patch \(\vec{X}\) in \(M\) (notice \(f \circ \vec{X}\) maps \(E^2\) [or more generally \(E^n\)] to \(\Omega\) and then to \(\mathbb{R}\) - so the idea of differentiability is clearly defined). For \(f : \Omega \rightarrow \mathbb{R}\) smooth and \(\vec{X} : D \rightarrow M\) an admissible patch whose image intersects \(\Omega\), define

\[
\frac{\partial f}{\partial u^i} : \vec{X}(D) \cap \Omega \rightarrow \mathbb{R} \text{ for } i = 1, 2 \text{ (or generally } i = 1, 2, \ldots, n) \]

as

\[
\frac{\partial f}{\partial u^i} = \frac{\partial (f \circ \vec{X})}{\partial u^i} \circ \vec{X}^{-1}.
\]

This is called the partial derivative of \(f\) with respect to \(u^i\).

Note. For \(\vec{P} \in \vec{X}(D) \cap \Omega\):

\[
\frac{\partial f}{\partial u^i}(\vec{P}) = \frac{\partial (f \circ \vec{X})}{\partial u^i}(\vec{X}^{-1}(\vec{P})).
\]
1.9. Manifolds

The mappings are:

\[ \tilde{P} \in M \xrightarrow{\tilde{X}^{-1}} \tilde{X}^{-1}(u^1, u^2) \xrightarrow{\partial_i (\partial \tilde{X})} \mathbb{R}. \]

The usual product rules hold:

\[ \frac{\partial}{\partial u^i} (fg) = \frac{\partial f}{\partial u^i} g + f \frac{\partial g}{\partial u^i} \]

where \( f \) and \( g \) have common domain.

**Definition.** For \( \tilde{P} \in \tilde{X}(D) \), define an operator on the collection of functions smooth in a neighborhood of \( \tilde{P} \) as

\[ \frac{\partial}{\partial u^i}(\tilde{P})[f] = \frac{\partial f}{\partial u^i}(\tilde{P}). \]

**Notation.** A superscript which appears in the denominator, such as \( \partial/\partial u^i \), counts as a subscript and therefore will impact the Einstein summation notation. (The motivation is that partial differentiation is usually denoted with subscripts.)

**Note.** If \( \tilde{X} : D \to M \) and \( \tilde{X} : \tilde{D} \to M \) are admissible patches, then on the overlap \( \tilde{X}(D) \cap \tilde{X}(\tilde{D}) \) we have from equation (61), page 83, the operator identities

\[ \frac{\partial}{\partial u^i} = \frac{\partial u^i}{\partial u^j} \frac{\partial}{\partial u^j} \text{ for } i = 1, 2 \]  \hspace{1cm} (63a)

\[ \frac{\partial}{\partial u^i} = \frac{\partial u^i}{\partial u^k} \frac{\partial}{\partial u^k} \text{ for } k = 1, 2 \]  \hspace{1cm} (63b)
Definition I-14. Let $m \in \mathbb{N}$ and suppose $O$ is an open subset of $E^m$. A function $f : O \rightarrow M$ is smooth if $\tilde{X}^{-1} \circ f$ (which maps $E^m$ to $E^n$) is smooth for every admissible patch $\tilde{X}$ on $M$. If $O$ is not open, we say $f : O \rightarrow M$ is smooth if $f$ is smooth on an open set containing $O$. A curve in $M$ is a smooth function from an interval (a connected subset of $\mathbb{R}$) into $M$.

Note. Now for tangent vectors and planes. We replace the idea of vectors as arrows, with the idea of vectors as operators. Remember that a vector is something which satisfies the properties given in the definition of a vector space! The “arrows” idea is just (technically) an aid in visualization!

Definition I-15. Let $\vec{\alpha} : I \rightarrow M$ be a curve on a 2-manifold (or generally, $n$-manifold) $M$. For $t \in I$, define the velocity vector of $\vec{\alpha}$ at $\vec{\alpha}(t)$ as the operator

$$\vec{\alpha}'(t)[f] = (f \circ \vec{\alpha})'(t) = \frac{d}{dt}[f(\vec{\alpha}(t))]$$

for each smooth $f$ which maps an open neighborhood of $\vec{\alpha}(t)$ into $\mathbb{R}$.

Definition I-16. Let $\vec{P}$ be a point of the 2-manifold $M$. An operator $\vec{v}$ which assigns a real number $\vec{v}[f]$ to each smooth real-valued function $f$ on $M$ is called a tangent vector to $M$ at $\vec{P}$ if there exists a curve in $M$ which passes through $\vec{P}$ and has velocity $\vec{v}$ at $\vec{P}$. The set of all tangent vectors to $M$ at $\vec{P}$ is called the tangent plane of $M$ at $\vec{P}$, denoted $T_{\vec{P}}M$.

Note. The previous two definitions are independent of the choice of coordinate patch (although we may do computations in some coordinate patch).
**Theorem.** Let \( \vec{P} \) be a point on manifold \( M \) and let \( \vec{X} \) be an admissible coordinate patch such that \( \vec{P} = \vec{X}(u^1(t_0), u^2(t_0)) \). If \( \vec{v} \) is a tangent vector to \( M \) at \( \vec{P} \) then \( \vec{v} \) is a linear combination of \( \frac{\partial}{\partial u^1}(\vec{P}) \) and \( \frac{\partial}{\partial u^2}(\vec{P}) \).

**Proof.** With \( \vec{v} \) a tangent vector, there is a curve \( \vec{\alpha}(t) \) in \( M \) such that \( \vec{\alpha}(t_0) = \vec{P} \) and \( \vec{\alpha}'(t_0) = \vec{v} \). Let \( f \) be a smooth real-valued function. Then with \( \vec{\alpha}(t) = \vec{X}(u^1(t), u^2(t)) \),

\[
\vec{v}(t)[f] = \vec{\alpha}'(t)[f] = \frac{d}{dt}[(f \circ \vec{\alpha})(t)] \\
= \frac{d}{dt}[f \circ \vec{X}(u^1(t), u^2(t))] = \frac{\partial(f \circ \vec{X})}{\partial u^1}(u^1(t), u^2(t)) \frac{du^i}{dt} \\
= \frac{\partial(f \circ \vec{X})}{\partial u^i}(\vec{X}^{-1} \circ \vec{\alpha}(t)) \frac{du^i}{dt} = \frac{\partial f}{\partial u^i}(\vec{\alpha}(t)) u^i(t) \text{ (by definition)} \\
= u^i(t) \frac{\partial f}{\partial u^i}(\vec{\alpha}(t)).
\]

So as an operator, \( \vec{\alpha}'(t) = u^i(t) \frac{\partial}{\partial u^i}(\vec{\alpha}(t)) \), or simply

\[
\vec{\alpha}' = u^i \frac{\partial}{\partial u^i}. \quad (64)
\]

At point \( \vec{P} \),

\[
\vec{v} = \vec{\alpha}'(t_0) = u^i(t_0) \frac{\partial}{\partial u^i}(\vec{P}) = v^i \frac{\partial}{\partial u^i}(\vec{P})
\]

where \( v^i = u^i(t_0) \).

**Note.** The vector \( \frac{\partial}{\partial u^1}(\vec{P}) \) and \( \frac{\partial}{\partial u^2}(\vec{P}) \) are linearly independent (consider their behavior on functions of the form \( f(u^1, u^2) = u^1 \) and \( g(u^1, u^2) = u^2 \)... although this argument is weak!). So the vectors form a basis for a 2-dimensional vector space, the tangent plane to \( M \) at \( \vec{P} \), \( T_{\vec{P}}M \). In general, a tangent plane to an \( n \)-manifold is an \( n \)-dimensional vector space (a “hyperplane”).
Note. The converse of the Theorem also holds: If $\vec{v}$ is a linear combination of $\frac{\partial}{\partial u^1}(\vec{P})$ and $\frac{\partial}{\partial u^2}(\vec{P})$, then $\vec{v}$ is a tangent vector to $M$ at $\vec{P}$.

Note. Suppose $\vec{X} : D \to M$ and $\vec{X} : \overline{D} \to M$ are overlapping admissible patches at $\vec{P}$. Then tangent vector $\vec{v}$ has two coordinate representations:

$$\vec{v} = v^i \frac{\partial}{\partial u^i}(\vec{P}) = \overline{v}^j \frac{\partial}{\partial \overline{u}^j}(\vec{P}).$$

From equation (63a), page 85, we have

$$\frac{\partial}{\partial u^i} = \frac{\partial \overline{u}^j}{\partial u^i} \frac{\partial}{\partial \overline{u}^j} \text{ for } i = 1, 2$$

and so

$$\vec{v} = v^i \frac{\partial}{\partial u^i}(\vec{P}) = v^i \left( \frac{\partial \overline{u}^j}{\partial u^i} \frac{\partial}{\partial \overline{u}^j} \right)(\vec{P}) = \left( v^i \frac{\partial \overline{u}^j}{\partial u^i} \right) \frac{\partial}{\partial \overline{u}^j}(\vec{P})$$

and so

$$\overline{v}^j = v^i \frac{\partial \overline{u}^j}{\partial u^i} \text{ for } j = 1, 2 \quad (67a)$$

(remember the linear independence of the $\partial/\partial \overline{u}^j$'s). Similarly

$$v^i = \overline{v}^j \frac{\partial}{\partial \overline{u}^j}(\vec{P}) \text{ for } i = 1, 2.$$

This gives us a relationship between the coordinates of tangent vectors. Notice that all these ideas extend to higher dimensions.

Note. We now introduce an inner product which generalizes the idea of a dot product and use this to carry over several of the ideas developed earlier for surfaces to manifolds.
**Definition I-17.** Let \( \mathcal{V} \) be a vector space with scalar field \( \mathbb{R} \). An *inner product* on \( \mathcal{V} \) is a mapping \( \langle \cdot , \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R} \) such that for all \( \vec{v}, \vec{v}', \vec{w}, \vec{w}' \in \mathcal{V} \) and for all \( a, a' \in \mathbb{R} \):

(a) \( \langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle \) (symmetry).

(b) \( \langle a\vec{v} + a'\vec{v}', \vec{w} \rangle = a\langle \vec{v}, \vec{w} \rangle + a'\langle \vec{v}', \vec{w} \rangle \) and \( \langle \vec{v}, a\vec{w} + a'\vec{w}' \rangle = a\langle \vec{v}, \vec{w} \rangle + a'\langle \vec{v}, \vec{w}' \rangle \) (bilinear).

(c) \( \langle \vec{v}, \vec{v} \rangle \geq 0 \) for all \( \vec{v} \in \mathcal{V} \) and \( \langle \vec{v}, \vec{v} \rangle = 0 \) if and only if \( \vec{v} = \vec{0} \) (positive definite).

**Definition I-18.** A *Riemannian metric* (or simply *metric*) on an \( 2 \)-manifold \( M \) is an assignment of an inner product to each tangent plane of \( M \). For each coordinate patch \( \vec{X} : D \rightarrow M \), we require the functions \( g_{ij} : \vec{X}(D) \rightarrow \mathbb{R} \) defined as

\[
g_{ij}(\vec{P}) = \left\langle \frac{\partial}{\partial u^i}(\vec{P}), \frac{\partial}{\partial u^j}(\vec{P}) \right\rangle
\]

for \( i, j = 1, 2, \ldots, n \) to be smooth. An \( n \)-manifold with such a Riemannian metric is called a *Riemannian \( n \)-manifold*.

**Example.** \( \mathbb{R}^n \) is a Riemannian manifold where the tangent planes are themselves \( \mathbb{R}^n \) (since \( \mathbb{R}^n \) is “flat”) and the inner product is the usual dot product in \( \mathbb{R}^n \).

**Example.** All the surfaces we dealt with earlier are examples of Riemannian \( 2 \)-manifolds (well... technically, a manifold does not have a boundary, so we might have to throw out some of the examples [such as the pseudosphere], although we could include in a study the so called “manifolds with a boundary”).
**Definition.** A vector space $\mathcal{V}$ with a mapping $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ satisfying (a) and (b) given above along with

(c') If $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{w} \in \mathcal{V}$, then $\vec{v} = \vec{0}$ (nonsingular).

is a \textit{semi-Riemannian} $n$–manifold (again, we require $g_{ij}(\vec{P})$ to be smooth).

**Note.** Condition (c') is weaker than condition (c) (and so every Riemannian $n$–manifold is also a semi-Riemannian $n$–manifold). Condition (c') allows lengths of vectors to be negative. We will see that spacetime is a semi-Riemannian 4-manifold.

**Note.** If $\vec{X} : D \to M$ and $\vec{X} : \overline{D} \to M$ are overlapping admissible patches then

$$\bar{g}_{mn} = g_{ij} \frac{\partial u^i}{\partial \bar{u}^m} \frac{\partial u^j}{\partial \bar{u}^n} \quad \text{for} \quad m, n = 1, 2,$$

$$g_{ij} = \bar{g}_{mn} \frac{\partial \bar{u}^m}{\partial u^i} \frac{\partial \bar{u}^n}{\partial u^j} \quad \text{for} \quad i, j = 1, 2.$$  

(You will verify these as homework.)

**Theorem.** If $\vec{v}$ and $\vec{w}$ are tangent vectors at $\vec{P}$ to a semi-Riemannian $n$–manifold $M$, and if $\vec{X} : D \to M$, $\vec{X} : \overline{D} \to M$ are admissible patches with $\vec{P} \in \vec{X}(D) \cap \vec{X}(\overline{D})$ then

$$g_{ij} v^i w^j = \bar{g}_{ij} \vec{v}^\bar{i} \vec{w}^\bar{j}.$$  

Therefore $g_{ij} v^i w^j$ is called an \textit{invariant}.

**Proof.** We have $\vec{v} = v^i \frac{\partial}{\partial u^i}$ and $\vec{w} = w^j \frac{\partial}{\partial u^j}$, so

$$\langle \vec{v}, \vec{w} \rangle = \left\langle v^i \frac{\partial}{\partial u^i}, w^j \frac{\partial}{\partial u^j} \right\rangle = v^i \left\langle \frac{\partial}{\partial u^i}, w^j \frac{\partial}{\partial u^j} \right\rangle = v^i w^j \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = g_{ij} v^i w^j.$$
Similarly, with \( \vec{v} = \vec{v}^i \frac{\partial}{\partial u^i} \) and \( \vec{w} = \vec{w}^j \frac{\partial}{\partial u^j} \) we have \( \langle \vec{v}, \vec{w} \rangle = g_{ij} \vec{v}^i \vec{w}^j \). Therefore \( g_{ij} v^i w^j = g_{ij} \vec{v}^i \vec{w}^j \). (This is consistent with the fact that inner products are independent of the choice of coordinates).

**Note.** We see from the above theorem, that the \( g_{ij} \)'s determine inner products of tangent vectors to a manifold just as the \( g_{ij} \)'s of Section 1.4 determined dot products of tangent vectors to a surface.

**Definition.** Let \( \vec{v} \) be a tangent vector to a semi-Riemannian \( n \)-manifold. Then define \( \| \vec{v} \| = \langle \vec{v}, \vec{v} \rangle^{1/2} \). For \( \bar{\alpha}(t), a \leq t \leq b \) a curve in \( M \), define the arclength of \( \bar{\alpha} \) as

\[
L = \int_a^b \| \bar{\alpha}'(t) \| \, dt.
\]

**Note.** Let \( s(t) = s \) denote the arc length along the curve from \( \bar{\alpha}(a) \) to \( \bar{\alpha}(t) \). Then

\[
s(t) = \int_a^t \| \bar{\alpha}'(t^*) \| \, dt^*
\]

and so \( s'(t) = \| \bar{\alpha}'(t) \| \) and

\[
(s'(t))^2 = \left( \frac{ds}{dt} \right)^2 = \| \bar{\alpha}'(t) \|^2 = \langle \bar{\alpha}', \bar{\alpha}'(t) \rangle.
\]

Let \( \vec{X} : D \to M \) be an admissible coordinate patch defined in a neighborhood of \( \bar{\alpha}(t) \). Then \( \bar{\alpha}' = \alpha^i \frac{\partial}{\partial u^i} = u^\nu \frac{\partial}{\partial u^i} \) (by equation (64), page 86) and as in the above Theorem

\[
\langle \bar{\alpha}'(t), \bar{\alpha}'(t) \rangle = \left\langle u^\nu \frac{\partial}{\partial u^i}, u^\nu \frac{\partial}{\partial u^j} \right\rangle = u^\nu u^\nu \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = g_{ij} u^\nu u^\nu = g_{ij} \left( \frac{du^i}{dt} \right) \left( \frac{dw^j}{dt} \right).
\]

(71)
1.9. Manifolds

Since expressions of the form $g_{ij}v^i w^j$ are invariant from one coordinate system to another, arclength and expression (71) are invariant.

**Definition.** Let $M$ be a semi-Riemannian manifold. The expression

$$\left(\frac{ds}{dt}\right)^2 = g_{ij}\frac{du^i}{dt} \frac{du^j}{dt}$$

(which is invariant from one “coordinate patch” to another) is the *metric form* or the *fundamental form* of the manifold.

**Note.** We now mimic earlier sections and give a number of definitions.

**Definition.** Create the matrix $(g_{ij})$ and define $(g_{ij})^{-1} = (g^{ij})$. For each coordinate system, $\vec{X}(u^1, u^2, \ldots, u^n)$ define the *Christoffel symbols of the first kind* as

$$\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

and the *Christoffel symbols of the second kind* as

$$\Gamma_{ij}^r = \frac{1}{2} g^{kr} \left( \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right).$$

**Definition I-19.** If $\vec{\alpha} = \vec{\alpha}(s)$ is a curve in a semi-Riemannian $n$–manifold $M$, where $s$ is arclength, then $\vec{\alpha}$ is a *geodesic* if in each local coordinate system defined on part of $\vec{\alpha}$

$$\frac{d^2 u^r}{ds^2} + \Gamma_{ij}^r \frac{du^i}{ds} \frac{du^j}{ds} = 0$$

for $r = 1, 2, \ldots, n$. (compare this to equation (29), page 58.)
Note. Theorems I-9 and I-10 carry over to semi-Riemannian $n$–manifolds. In particular, the shortest distance between two points is along a geodesic.

Definition. For a semi-Riemannian $n$–manifold, define the Riemann-Christoffel curvature tensor as

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial u^j} - \frac{\partial \Gamma^h_{ij}}{\partial u^k} + \Gamma^r_{ik} \Gamma^h_{jr} - \Gamma^r_{ij} \Gamma^h_{rk}$$

for $h, i, j, k = 1, 2, \ldots, n$. Define

$$R_{mijk} = g_{mh} R^h_{ijk}.$$

Note. The curvature tensor has $n^4$ entries (although there is some symmetry). When $n = 2$ the only nonzero entries are

$$R_{1212} = R_{2121} = -R_{2112} = -R_{1221}$$

and for 2-manifolds (as in Section 1.8), curvature is $K = R_{1212}/g$. However, things are much more complicated in higher dimensions!

Note. The curvature tensor $R^h_{ijk}$ for an $n$–manifold has $n^2(n^2-1)/12$ independent components (so sayeth the text, page 90). Therefore curvature for an $n$–manifold is NOT determined by a single number when $n > 2$!
Example (Exercise 1.9.4). Suppose a Riemannian metric on $M$ (an open subset of $\mathbb{R}^2$) is given by

$$ds^2 = \frac{1}{\gamma^2}(du^2 + dv^2)$$

where $\gamma = \gamma(u, v)$ is a smooth positive-valued function. Then $M$ has Gauss curvature

$$K = \gamma(\gamma_{uu} + \gamma_{vv}) - (\gamma_u^2 + \gamma_v^2).$$

Proof. First, we have $E = 1/\gamma^2 = G$ and $F = 0$. So we have from Exercise 1.8.3

$$K = \frac{-1}{\sqrt{EG}} \left\{ \frac{\partial}{\partial u} \left[ \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right] + \frac{\partial}{\partial v} \left[ \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right] \right\}.$$

Now $\sqrt{E} = \sqrt{G} = 1/\gamma$ and so

$$K = -\gamma \gamma \left\{ \frac{\partial}{\partial u} \left[ \gamma \frac{\partial [1/\gamma]}{\partial u} \right] + \frac{\partial}{\partial v} \left[ \gamma \frac{\partial [1/\gamma]}{\partial v} \right] \right\}$$

$$= -\gamma^2 \left\{ \frac{\partial}{\partial u} \left[ \gamma \frac{-1}{\gamma} \gamma_u \right] + \frac{\partial}{\partial v} \left[ \gamma \frac{-1}{\gamma^2} \gamma_v \right] \right\}$$

$$= -\gamma^2 \left\{ \frac{\partial}{\partial u} \left[ \frac{-\gamma_u}{\gamma} \right] + \frac{\partial}{\partial v} \left[ \frac{-\gamma_v}{\gamma} \right] \right\}$$

$$= -\gamma^2 \left\{ \frac{(-\gamma_{uu})\gamma - (-\gamma_u)(\gamma_u)}{\gamma^2} + \frac{(-\gamma_{vv})\gamma - (-\gamma_v)(\gamma_v)}{\gamma^2} \right\}$$

$$= -\gamma^2 \left\{ \frac{-\gamma_{uu} + (\gamma_u)^2 - \gamma_{vv} + (\gamma_v)^2}{\gamma^2} \right\}$$

$$= \gamma(\gamma_{uu} + \gamma_{vv}) - ((\gamma_u)^2 + (\gamma_v)^2).$$

\[\blacksquare\]
Example (Exercise 1.9.7). Let $M$ be the subset of $\mathbb{R}^2$: $M = \{(u, v) \mid u^2 + v^2 < 4k^2\}$ (where $k > 0$). Introduce the metric

$$ds^2 = \frac{1}{\gamma^2}(du^2 + dv^2)$$

where $\gamma(u, v) = 1 - \frac{u^2 + v^2}{4k^2}$. This is called the Poincare Disk. Then $K = -1/k^2$.

Proof. From Exercise 1.9.4,

$$K = \gamma(\gamma_{uu} + \gamma_{vv}) - (\gamma_u^2 + \gamma_v^2).$$

Well,

$$\gamma_u = \frac{-u}{2k^2}, \gamma_v = \frac{-v}{2k^2}, \gamma_{uu} = \frac{-1}{2k^2}, \gamma_{vv} = \frac{-1}{2k^2}.$$

Therefore,

$$K = \left(1 - \frac{u^2 + v^2}{4k^2}\right)\left(\frac{-1}{2k^2} + \frac{-1}{2k^2}\right) - \left(\frac{-u}{2k^2}\right)^2 + \left(\frac{-v}{2k^2}\right)^2.$$
\[\begin{align*}
\ &= \left(1 - \frac{u^2 + v^2}{4k^2}\right)\left(-\frac{1}{k^2}\right) - \left(\left(\frac{u^2}{4k^4}\right)^2 + \left(\frac{v^2}{4k^4}\right)^2\right) \\
\ &= -\frac{(4k^2 - u^2 - v^2)}{4k^4} - \frac{u^2}{4k^4} - \frac{v^2}{4k^4} = -\frac{1}{k^2}.
\end{align*}\]

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