1.9 Manifolds

Note. In this section, we extend the ideas of tangents, metrics, geodesics, and curvature to "manifolds" (in a sense, "n−dimensional surfaces") without appealing to how they are embedded in a higher dimensional space.

Definition. Let M be a non-empty set whose elements we call *points*. A *coordinate patch* on M is a one-to-one function $\vec{X} : D \to M$ (continuous and regular) from an open subset D of E^2 (or more generally E^n) into M.

Note. In the following definition, by "domain" of a function we mean the largest set on which the function is defined. By "smooth" we mean sufficiently differentiable for our purposes. A function whose domain is empty is considered smooth.

Definition I-12. An *abstract surface* or 2−*manifold* (more generally, n−*manifold*) is a set M with a collection $\mathcal{C}$ of coordinate patches on M satisfying:

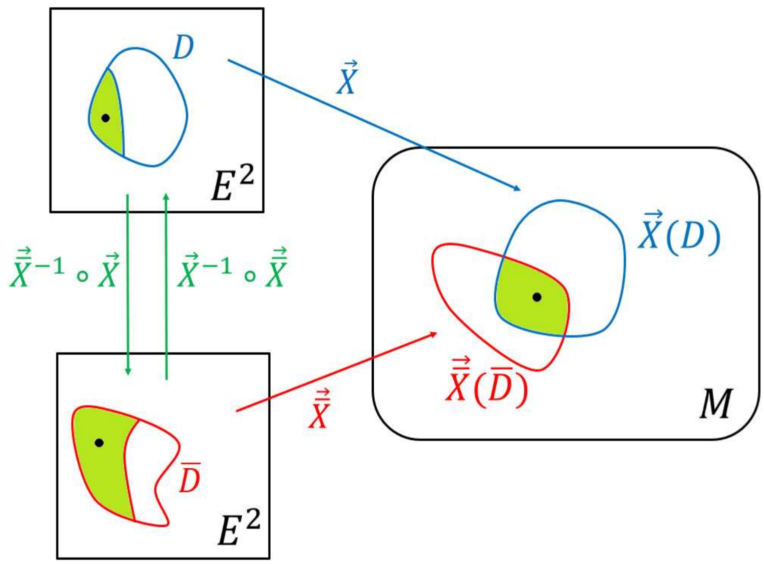
(a) M is the union of images of the patches in $\mathcal{C}$ (that is, if $\mathcal{C} = \{\vec{X}^i\}$ and $\vec{X}^i$ is defined on set D^i, then $M = \bigcup_i \vec{X}^i(D^i)$).

(b) The patches of $\mathcal{C}$ *overlap smoothly*, that is if $\vec{X}^1 : D^1 \to M$ and $\vec{X}^2 : D^2 \to M$ are two patches in $\mathcal{C}$, then $(\vec{X}^1)^{-1} \circ \vec{X}^2$ and $(\vec{X}^2)^{-1} \circ \vec{X}^1$ have open domains and are smooth.

(c) Given two points $\vec{P}^1$ and $\vec{P}^2$ of M, there exist coordinate patches $\vec{X}^1 : D^1 \to M$ and $\vec{X}^2 : D^2 \to M$ in $\mathcal{C}$ such that $\vec{P}^1 \in \vec{X}^1(D^1)$, $\vec{P}^2 \in \vec{X}^2(D^2)$ and

$\vec{X}^1(D^1) \cap \vec{X}^2(D^2) = \emptyset$ (this is the *Hausdorff property*).

(d) The collection $\mathcal{C}$ is *maximal*. That is, any coordinate patch on M which overlaps smoothly with every patch of $\mathcal{C}$ is itself in $\mathcal{C}$. (Notice that two disjoint coordinate patches "overlap smoothly" by convention).



Definition. The collection $\mathcal{C}$ is called a *differentiable structure* on M and patches in $\mathcal{C}$ are called *admissible patches.*

Note. If properties (a), (b), and (c) of Definition I-12 are satisfied by a collection $\mathcal{C}'$ then we can adjoin to $\mathcal{C}'$ all patches that overlap smoothly with the patches of $\mathcal{C}'$ to create a collection $\mathcal{C}$ which satisfies (a), (b), (c), (d). In this case, $\mathcal{C}'$ is said to *generate* $\mathcal{C}$.

Example (Exercise 1.9.1). Let M be the plane with Cartesian coordinates. The identity mapping of M onto itself is a coordinate patch. A differentiable structure on M is obtained by adjoining to this mapping all patches in M which overlap smoothly with this mapping. The polar coordinate patch

$$u = r\cos\theta \qquad v = r\sin\theta \qquad (r,\theta) \in D$$

overlaps smoothly with the identity patch IF D is of the form

$$D = \{(r,\theta) \mid r > 0, \theta \in (a,b), b - a \leq 2\pi\}$$

(an open sector).

Solution. Let $\vec{X} : D \to M$ and $\vec{\overline{X}} : \overline{D} \to M$. By definition, $\vec{X}$ and $\vec{\overline{X}}$ overlap smoothly if $(\vec{X})^{-1} \circ \vec{\overline{X}}$ (which maps $\overline{D} \to M \to D$) and $(\vec{\overline{X}})^{-1} \circ \vec{X}$ (which maps $D \to M \to \overline{D}$) have open domains and are smooth.

First, $\vec{X}$ and $\vec{\overline{X}}$ are one-to-one and so are invertible. Explicitly,

$$(\vec{X})^{-1} \circ \vec{\overline{X}}(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta).$$

So

$$\frac{\partial}{\partial r}[(\vec{X})^{-1} \circ \vec{\overline{X}}] = (\cos\theta, \sin\theta)$$

and

$$\frac{\partial}{\partial \theta}[(\vec{X})^{-1} \circ \vec{\overline{X}}] = (-r\sin\theta, r\cos\theta).$$

Therefore, $(\vec{X})^{-1} \circ \vec{\overline{X}}$ is smooth (the first partials are continuous... in fact, it is infinitely differentiable). Similarly, $(\vec{\overline{X}})^{-1} \circ \vec{X}(x,y) = (r,\theta)$ where $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$ where we choose θ such that $\theta \in (a,b)$,

θ is in Quadrant I if $x > 0$, $y > 0$,

θ is in Quadrant II if $x < 0$, $y > 0$,

θ is in Quadrant III if $x < 0$, $y < 0$,

θ is in Quadrant IV if $x > 0$, $y < 0$,

(and similar choices are made if $x = 0$ or $y = 0$). So $\theta = \tan^{-1}(y/x) + \text{constant}_\theta$ (so θ is a continuous function of (x, y), even though $\tan^{-1}(y/x)$ is not continuous — this is how we choose the θ to associate with (x, y)). We then have

$$\frac{\partial}{\partial x}[(\vec{\overline{X}}^{-1} \circ \vec{X})] = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{-y/x^2}{1+(y/x)^2}\right)$$

and

$$\frac{\partial}{\partial y}[(\vec{\overline{X}}^{-1} \circ \vec{X})] = \left(\frac{y}{\sqrt{x^2+y^2}}, \frac{1/x}{1+(y/x)^2}\right),$$

therefore (since $(x, y) \neq (0, 0)$) $(\vec{\overline{X}})^{-1} \circ \vec{X}$ is smooth (in fact, infinitely differentiable). Next, the domain of $(\vec{\overline{X}})^{-1} \circ \vec{X} : D \to \overline{D}$ is D itself and D is open (by definition). The domain of $(\vec{X})^{-1} \circ \vec{\overline{X}} : \overline{D} \to D$ is the set of all $(x, y) \in M$ such that $\sqrt{x^2+y^2} = r > 0$ and $\tan^{-1}(y/x) \in (a, b)$ (where $\tan^{-1}(y/x)$ is calculated as described above). Therefore the domain of $(\vec{\overline{X}})^{-1} \circ \vec{X}$ is open. Hence, $\vec{X}$ and $\vec{\overline{X}}$ overlap smoothly.

Definition. An admissible patch $\vec{X} : D \to M$ associates with each point $\vec{P}$ of $\vec{X}(D)$ a unique ordered pair (or in general, ordered $n-$tuple) $(u^1, u^2) = \vec{X}^{-1}(\vec{P})$ called a *local coordinate* of $\vec{P}$ with respect to $\vec{X}$.

Note. A point $\vec{P}$ can have different local coordinates with respect to different admissible patches. Suppose, for example, $\vec{P} = \vec{X}(u^1, u^2) = \vec{\overline{X}}(\overline{u}^1, \overline{u}^2)$. Then $(\vec{X})^{-1} \circ \vec{\overline{X}}(\overline{u}^1, \overline{u}^2) = (u^1, u^2)$ and $(\vec{\overline{X}})^{-1} \circ \vec{X}^1(u^1, u^2) = (\overline{u}^1, \overline{u}^2)$.

Definition. In the above setting, the equations $(\vec{X})^{-1} \circ \vec{\overline{X}}(\overline{u}^1, \overline{u}^2) = (u^1, u^2)$ and $(\vec{\overline{X}})^{-1} \circ \vec{X}(u^1, u^1) = (\overline{u}^1, \overline{u}^2)$ are *changes of coordinates*. See Figure I-29, page 83 (a form of which is above, after the definition of 2-*manifold*). In terms of local coordinates:

$$\vec{\overline{X}}^{-1} \circ \vec{X} \text{ is given by } \overline{u}^i = \overline{u}^i(u^1, u^2),\ i = 1, 2 \qquad (61a)$$

$$\vec{X}^{-1} \circ \vec{\overline{X}} \text{ is given by } u^i = u^i(\overline{u}^1, \overline{u}^2),\ i = 1, 2. \qquad (61b)$$

Definition I-13. A set $\Omega \subset M$ is a *neighborhood* of a point $\vec{P} \in M$ if there exists an admissible patch $\vec{X} : D \to M$ such that $\vec{P} \in \vec{X}(D)$ and $\vec{X}(D) \subset \Omega$. A subset of M is *open* if it is a neighborhood of each of its points.

Definition. Let Ω be an open subset of the 2-manifold (or generally $n-$manifold) M. A function $f : \Omega \to \mathbb{R}$ is *smooth* if $f \circ \vec{X}$ is smooth for every admissible patch $\vec{X}$ in M (notice $f \circ \vec{X}$ maps E^2 [or more generally E^n] to Ω and then to $\mathbb{R}$ - so the idea of differentiability is clearly defined). For $f : \Omega \to \mathbb{R}$ smooth and $\vec{X} : D \to M$ an admissible patch whose image intersects Ω, define

$$\frac{\partial f}{\partial u^i} : \vec{X}(D) \cap \Omega \to \mathbb{R} \text{ for } i = 1, 2 \text{ (or generally } i = 1, 2, \ldots, n)$$

as

$$\frac{\partial f}{\partial u^i} = \frac{\partial (f \circ \vec{X})}{\partial u^i} \circ \vec{X}^{-1}.$$

This is called the *partial derivative of f with respect to u^i*.

Note. For $\vec{P} \in \vec{X}(D) \cap \Omega$:

$$\frac{\partial f}{\partial u^i}(\vec{P}) = \frac{\partial (f \circ \vec{X})}{\partial u^i}(\vec{X}^{-1}(\vec{P})).$$

The mappings are:

$$\vec{P} \in M \xrightarrow{\vec{X}^{-1}} \vec{X}^{-1}(u^1, u^2) \xrightarrow{\frac{\partial(f \circ \vec{X})}{\partial u^i}} \mathbb{R}.$$

The usual product rules hold:

$$\frac{\partial}{\partial u^i}(fg) = \frac{\partial f}{\partial u^i} g + f \frac{\partial g}{\partial u^i}$$

where f and g have common domain.

Definition. For $\vec{P} \in \vec{X}(D)$, define an operator on the collection of functions smooth in a neighborhood of $\vec{P}$ as

$$\frac{\partial}{\partial u^i}(\vec{P})[f] = \frac{\partial f}{\partial u^i}(\vec{P}).$$

Notation. A superscript which appears in the denominator, such as $\partial/\partial u^i$, counts as a subscript and therefore will impact the Einstein summation notation. (The motivation is that partial differentiation is usually denoted with subscripts.)

Note. If $\vec{X} : D \to M$ and $\overline{\vec{X}} : \overline{D} \to M$ are admissible patches, then on the overlap $\vec{X}(D) \cap \overline{\vec{X}}(\overline{D})$ we have from equation (61), page 83, the operator identities

$$\frac{\partial}{\partial u^i} = \frac{\partial \overline{u}^j}{\partial u^i} \frac{\partial}{\partial \overline{u}^j} \text{ for } i = 1, 2 \qquad (63a)$$

$$\frac{\partial}{\partial \overline{u}^k} = \frac{\partial u^i}{\partial \overline{u}^k} \frac{\partial}{\partial u^i} \text{ for } k = 1, 2 \qquad (63b)$$

Definition I-14. Let $m \in \mathbb{N}$ and suppose $\mathcal{O}$ is an open subset of E^m. A function $f : \mathcal{O} \to \mathcal{M}$ is *smooth* if $\vec{X}^{-1} \circ f$ (which maps E^m to E^n) is smooth for every admissible patch $\vec{X}$ on M. If $\mathcal{O}$ is not open, we say $f : \mathcal{O} \to \mathcal{M}$ is smooth if f is smooth on an open set containing $\mathcal{O}$. A *curve* in M is a smooth function from an interval (a connected subset of $\mathbb{R}$) into M.

Note. Now for tangent vectors and planes. We replace the idea of vectors as *arrows*, with the idea of vectors as *operators*. Remember that a vector is something which satisfies the properties given in the definition of a vector space! The "arrows" idea is just (technically) an aid in visualization!

Definition I-15. Let $\vec{\alpha} : I \to M$ be a curve on a 2-manifold (or generally, $n-$manifold) M. For $t \in I$, define the *velocity vector* of $\vec{\alpha}$ at $\vec{\alpha}(t)$ as the operator

$$\vec{\alpha}'(t)[f] = (f \circ \vec{\alpha})'(t) = \frac{d}{dt}[f(\vec{\alpha}(t))]$$

for each smooth f which maps an open neighborhood of $\vec{\alpha}(t)$ into $\mathbb{R}$.

Definition I-16. Let $\vec{P}$ be a point of the $2-$manifold M. An operator $\vec{v}$ which assigns a real number $\vec{v}[f]$ to each smooth real-valued function f on M is called a *tangent vector* to M at $\vec{P}$ if there exists a curve in M which passes through $\vec{P}$ and has velocity $\vec{v}$ at $\vec{P}$. The set of all tangent vectors to M at $\vec{P}$ is called the *tangent plane* of M at $\vec{P}$, denoted $T_{\vec{P}}M$.

Note. The previous two definitions are independent of the choice of coordinate patch (although we may do computations in some coordinate patch).

Theorem. Let $\vec{P}$ be a point on manifold M and let $\vec{X}$ be an admissible coordinate patch such that $\vec{P} = \vec{X}(u^1(t_0), u^2(t_0))$. If $\vec{v}$ is a tangent vector to M at $\vec{P}$ then $\vec{v}$ is a linear combination of $\frac{\partial}{\partial u^1}(\vec{P})$ and $\frac{\partial}{\partial u^2}(\vec{P})$.

Proof. With $\vec{v}$ a tangent vector, there is a curve $\vec{\alpha}(t)$ in M such that $\vec{\alpha}(t_0) = \vec{P}$ and $\vec{\alpha}'(t_0) = \vec{v}$. Let f be a smooth real-valued function. Then with $\vec{\alpha}(t) = \vec{X}(u^1(t), u^2(t))$,

$$\begin{aligned}
\vec{v}(t)[f] &= \vec{\alpha}'(t)[f] = \frac{d}{dt}[(f \circ \vec{\alpha})(t)] \\
&= \frac{d}{dt}[f \circ \vec{X}(u^1(t), u^2(t))] = \frac{\partial (f \circ \vec{X})}{\partial u^i}(u^1(t), u^2(t))\frac{du^i}{dt} \\
&= \frac{\partial (f \circ \vec{X})}{\partial u^i}(\vec{X}^{-1} \circ \vec{\alpha}(t))\frac{du^i}{dt} = \frac{\partial f}{\partial u^i}(\vec{\alpha}(t))u^{i\prime}(t) \text{ (by definition)} \\
&= u^{i\prime}(t)\frac{\partial f}{\partial u^i}(\vec{\alpha}(t)).
\end{aligned}$$

So as an operator, $\vec{\alpha}'(t) = u^{i\prime}(t)\frac{\partial}{\partial u^i}(\vec{\alpha}(t))$, or simply

$$\vec{\alpha}' = u^{i\prime}\frac{\partial}{\partial u^i}. \qquad (64)$$

At point $\vec{P}$,

$$\vec{v} = \vec{\alpha}'(t_0) = u^{i\prime}(t_0)\frac{\partial}{\partial u^i}(\vec{P}) = v^i\frac{\partial}{\partial u^i}(\vec{P})$$

where $v^i = u^{i\prime}(t_0)$. ∎

Note. The vector $\frac{\partial}{\partial u^1}(\vec{P})$ and $\frac{\partial}{\partial u^2}(\vec{P})$ are linearly independent (consider their behavior on functions of the form $f(u^1, u^2) = u^1$ and $g(u^1, u^2) = u^2$... although this argument is weak!). So the vectors form a basis for a 2-dimensional vector space, the tangent plane to M at $\vec{P}$, $T_{\vec{P}}M$. In general, a tangent plane to an $n-$manifold is an $n-$dimensional vector space (a "hyperplane").

Note. The converse of the Theorem also holds: If $\vec{v}$ is a linear combination of $\frac{\partial}{\partial u^1}(\vec{P})$ and $\frac{\partial}{\partial u^2}(\vec{P})$, then $\vec{v}$ is a tangent vector to M at $\vec{P}$.

Note. Suppose $\vec{X} : D \to M$ and $\vec{\overline{X}} : \overline{D} \to M$ are overlapping admissible patches at $\vec{P}$. Then tangent vector $\vec{v}$ has two coordinate representations:

$$\vec{v} = v^i \frac{\partial}{\partial u^i}(\vec{P}) = \overline{v}^j \frac{\partial}{\partial \overline{u}^j}(\vec{P}).$$

From equation (63a), page 85, we have

$$\frac{\partial}{\partial u^i} = \frac{\partial \overline{u}^j}{\partial u^i}\frac{\partial}{\partial \overline{u}^j} \text{ for } i = 1, 2$$

and so

$$\vec{v} = v^i \frac{\partial}{\partial u^i}(\vec{P}) = v^i \left(\frac{\partial \overline{u}^j}{\partial u^i}\frac{\partial}{\partial \overline{u}^j}\right)(\vec{P}) = \left(v^i \frac{\partial \overline{u}^j}{\partial u^i}\right)\frac{\partial}{\partial \overline{u}^j}(\vec{P})$$

and so

$$\overline{v}^j = v^i \frac{\partial \overline{u}^j}{\partial u^i} \text{ for } j = 1, 2 \qquad (67a)$$

(remember the linear independence of the $\partial/\partial\overline{u}^j$'s). Similarly

$$v^i = \overline{v}^j \frac{\partial}{\partial \overline{u}^j}(\vec{P}) \text{ for } i = 1, 2.$$

This gives us a relationship between the coordinates of tangent vectors. Notice that all these ideas extend to higher dimensions.

Note. We now introduce an inner product which generalizes the idea of a dot product and use this to carry over several of the ideas developed earlier for surfaces to manifolds.

Definition I-17. Let $\mathcal{V}$ be a vector space with scalar field $\mathbb{R}$. An *inner product* on $\mathcal{V}$ is a mapping $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that for all $\vec{v}, \vec{v}', \vec{w}, \vec{w}' \in \mathcal{V}$ and for all $a, a' \in \mathbb{R}$:

(a) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$ (symmetry).

(b) $\langle a\vec{v} + a'\vec{v}', \vec{w} \rangle = a\langle \vec{v}, \vec{w} \rangle + a'\langle \vec{v}', \vec{w} \rangle$ and $\langle \vec{v}, a\vec{w} + a'\vec{w}' \rangle = a\langle \vec{v}, \vec{w} \rangle + a'\langle \vec{v}, \vec{w}' \rangle$ (bilinear).

(c) $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v} \in \mathcal{V}$ and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$ (positive definite).

Definition I-18. A *Riemannian metric* (or simply *metric*) on an $2-$manifold M is an assignment of an inner product to each tangent plane of M. For each coordinate patch $\vec{X} : D \to M$, we require the functions $g_{ij} : \vec{X}(D) \to \mathbb{R}$ defined as

$$g_{ij}(\vec{P}) = \left\langle \frac{\partial}{\partial u^i}(\vec{P}), \frac{\partial}{\partial u^j}(\vec{P}) \right\rangle$$

for $i, j = 1, 2, \ldots, n$ to be smooth. An $n-$manifold with such a Riemannian metric is called a *Riemannian $n-$manifold.*

Example. $\mathbb{R}^n$ is a Riemannian manifold where the tangent planes are themselves $\mathbb{R}^n$ (since $\mathbb{R}^n$ is "flat") and the inner product is the usual dot product in $\mathbb{R}^n$.

Example. All the surfaces we dealt with earlier are examples of Riemannian 2-manifolds (well. . . technically, a manifold does not have a boundary, so we might have to throw out some of the examples [such as the pseudosphere], although we could include in a study the so called "manifolds with a boundary").

Definition. A vector space $\mathcal{V}$ with a mapping $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ satisfying (a) and (b) given above along with

(c′) If $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{w} \in \mathcal{V}$, then $\vec{v} = \vec{0}$ (nonsingular).

is a *semi-Riemannian* $n-$manifold (again, we require $g_{ij}(\vec{P})$ to be smooth).

Note. Condition (c′) is weaker than condition (c) (and so every Riemannian $n-$manifold is also a semi-Riemannian $n-$manifold). Condition (c′) allows lengths of vectors to be negative. We will see that spacetime is a semi-Riemannian 4-manifold.

Note. If $\vec{X} : D \to M$ and $\vec{\overline{X}} : \overline{D} \to M$ are overlapping admissible patches then

$$\overline{g}_{mn} = g_{ij} \frac{\partial u^i}{\partial \overline{u}^m} \frac{\partial u^j}{\partial \overline{u}^n} \text{ for } m, n = 1, 2,$$

$$g_{ij} = \overline{g}_{mn} \frac{\partial \overline{u}^m}{\partial u^i} \frac{\partial \overline{u}^n}{\partial u^j} \text{ for } i, j = 1, 2.$$

(You will verify these as homework.)

Theorem. If $\vec{v}$ and $\vec{w}$ are tangent vectors at $\vec{P}$ to a semi-Riemannian $n-$manifold M, and if $\vec{X} : D \to M$, $\vec{\overline{X}} : \overline{D} \to M$ are admissible patches with $\vec{P} \in \vec{X}(D) \cap \vec{\overline{X}}(\overline{D})$ then

$$g_{ij} v^i w^j = \overline{g}_{ij} \overline{v}^i \overline{w}^j.$$

Therefore $g_{ij} v^i w^j$ is called an *invariant*.

Proof. We have $\vec{v} = v^i \dfrac{\partial}{\partial u^i}$ and $\vec{w} = w^j \dfrac{\partial}{\partial u^j}$, so

$$\langle \vec{v}, \vec{w} \rangle = \left\langle v^i \frac{\partial}{\partial u^i}, w^j \frac{\partial}{\partial u^j} \right\rangle = v^i \left\langle \frac{\partial}{\partial u^i}, w^j \frac{\partial}{\partial u^j} \right\rangle = v^i w^j \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = g_{ij} v^i w^j.$$

Similarly, with $\vec{v} = \overline{v}^i \frac{\partial}{\partial \overline{u}^i}$ and $\vec{w} = \overline{w}^j \frac{\partial}{\partial \overline{u}^j}$ we have $\langle \vec{v}, \vec{w} \rangle = \overline{g}_{ij} \overline{v}^i \overline{w}^j$. Therefore $g_{ij} v^i w^j = \overline{g}_{ij} \overline{v}^i \overline{w}^j$. (This is consistent with the fact that inner products are independent of the choice of coordinates). ∎

Note. We see from the above theorem, that the g_{ij}'s determine inner products of tangent vectors to a manifold just as the g_{ij}'s of Section 1.4 determined dot products of tangent vectors to a surface.

Definition. Let $\vec{v}$ be a tangent vector to a semi-Riemannian $n-$manifold. Then define $\|\vec{v}\| = \langle \vec{v}, \vec{v} \rangle^{1/2}$. For $\vec{\alpha}(t)$, $a \le t \le b$ a curve in M, define the *arclength* of $\vec{\alpha}$ as

$$L = \int_a^b \|\vec{\alpha}'(t)\| \, dt.$$

Note. Let $s(t) = s$ denote the arc length along the curve from $\vec{\alpha}(a)$ to $\vec{\alpha}(t)$. Then

$$s(t) = \int_a^t \|\vec{\alpha}'(t^*)\| \, dt^*$$

and so $s'(t) = \|\vec{\alpha}'(t)\|$ and

$$(s'(t))^2 = \left(\frac{ds}{dt} \right)^2 = \|\vec{\alpha}'(t)\|^2 = \langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle.$$

Let $\vec{X} : D \to M$ be an admissible coordinate patch defined in a neighborhood of $\vec{\alpha}(t)$. Then $\vec{\alpha}' = \alpha^i \frac{\partial}{\partial u^i} = u^{i\prime} \frac{\partial}{\partial u^i}$ (by equation (64), page 86) and as in the above Theorem

$$\begin{aligned} \langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle &= \left\langle u^{i\prime} \frac{\partial}{\partial u^i}, u^{j\prime} \frac{\partial}{\partial u^j} \right\rangle = u^{i\prime} u^{j\prime} \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle \\ &= g_{ij} u^{i\prime} u^{j\prime} = g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}. \qquad (71) \end{aligned}$$

Since expressions of the form $g_{ij}v^iw^j$ are invariant from one coordinate system to another, arclength and expression (71) are invariant.

Definition. Let M be a semi-Riemannian manifold. The expression

$$\left(\frac{ds}{dt}\right)^2 = g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}$$

(which is invariant from one "coordinate patch" to another) is the *metric form* or the *fundamental form* of the manifold.

Note. We now mimic earlier sections and give a number of definitions.

Definition. Create the matrix (g_{ij}) and define $(g_{ij})^{-1} = (g^{ij})$. For each coordinate system, $\vec{X}(u^1, u^2, \ldots, u^n)$ define the *Christoffel symbols of the first kind* as

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)$$

and the *Christoffel symbols of the second kind* as

$$\Gamma^r_{ij} = \frac{1}{2}g^{kr}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right).$$

Definition I-19. If $\vec{\alpha} = \vec{\alpha}(s)$ is a curve in a semi-Riemannian $n-$manifold M, where s is arclength, then $\vec{\alpha}$ is a *geodesic* if in each local coordinate system defined on part of $\vec{\alpha}$

$$\frac{d^2u^r}{ds^2} + \Gamma^r_{ij}\frac{du^i}{ds}\frac{du^j}{ds} = 0$$

for $r = 1, 2, \ldots, n$. (compare this to equation (29), page 58.)

Note. Theorems I-9 and I-10 carry over to semi-Riemannian $n-$manifolds. In particular, the shortest distance between two points is along a geodesic.

Definition. For a semi-Riemannian $n-$manifold, define the *Riemann-Christoffel curvature tensor* as

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial u^j} - \frac{\partial \Gamma^h_{ij}}{\partial u^k} + \Gamma^r_{ik}\Gamma^h_{rj} - \Gamma^r_{ij}\Gamma^h_{rk}$$

for $h, i, j, k = 1, 2, \ldots, n$. Define

$$R_{mijk} = g_{mh} R^h_{ijk}.$$

Note. The curvature tensor has n^4 entries (although there is some symmetry). When $n = 2$ the only nonzero entries are

$$R_{1212} = R_{2121} = -R_{2112} = -R_{1221}$$

and for 2-manifolds (as in Section 1.8), curvature is $K = R_{1212}/g$. However, things are much more complicated in higher dimensions!

Note. The curvature tensor R^h_{ijk} for an $n-$manifold has $n^2(n^2-1)/12$ independent components (so sayeth the text, page 90). Therefore curvature for an $n-$manifold is NOT determined by a single number
when $n > 2$!

Example (Exercise 1.9.4). Suppose a Riemannian metric on M (an open subset of $\mathbb{R}^2$) is given by

$$ds^2 = \frac{1}{\gamma^2}(du^2 + dv^2)$$

where $\gamma = \gamma(u, v)$ is a smooth positive-valued function. Then M has Gauss curvature

$$K = \gamma(\gamma_{uu} + \gamma_{vv}) - (\gamma_u^2 + \gamma_v^2).$$

Proof. First, we have $E = 1/\gamma^2 = G$ and $F = 0$. So we have from Exercise 1.8.3

$$K = \frac{-1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right]\right\}.$$

Now $\sqrt{E} = \sqrt{G} = 1/\gamma$ and so

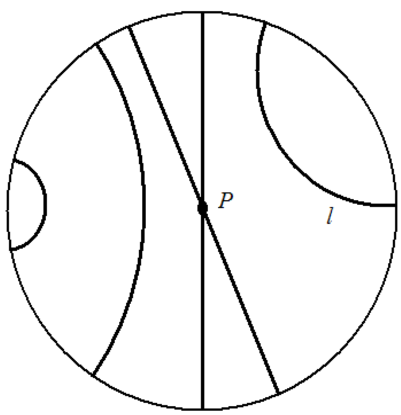
$$\begin{aligned}
K &= -\gamma\gamma\left\{\frac{\partial}{\partial u}\left[\gamma\frac{\partial[1/\gamma]}{\partial u}\right] + \frac{\partial}{\partial v}\left[\gamma\frac{\partial[1/\gamma]}{\partial v}\right]\right\} \\
&= -\gamma^2\left\{\frac{\partial}{\partial u}\left[\gamma\frac{-1}{\gamma}\gamma_u]\right] + \frac{\partial}{\partial v}\left[\gamma\frac{-1}{\gamma^2}\gamma_v\right]\right\} \\
&= -\gamma^2\left\{\frac{\partial}{\partial u}\left[\frac{-\gamma_u}{\gamma}\right] + \frac{\partial}{\partial v}\left[\frac{-\gamma_v}{\gamma}\right]\right\} \\
&= -\gamma^2\left\{\frac{(-\gamma_{uu})\gamma - (-\gamma_u)(\gamma_u)}{\gamma^2} + \frac{(-\gamma_{vv})\gamma - (-\gamma_v)(\gamma_v)}{\gamma^2}\right\} \\
&= -\gamma^2\left\{\frac{-\gamma\gamma_{uu} + (\gamma_u)^2 - \gamma\gamma_{vv} + (\gamma_v)^2}{\gamma^2}\right\} \\
&= \gamma(\gamma_{uu} + \gamma_{vv}) - ((\gamma_u)^2 + (\gamma_v)^2).
\end{aligned}$$

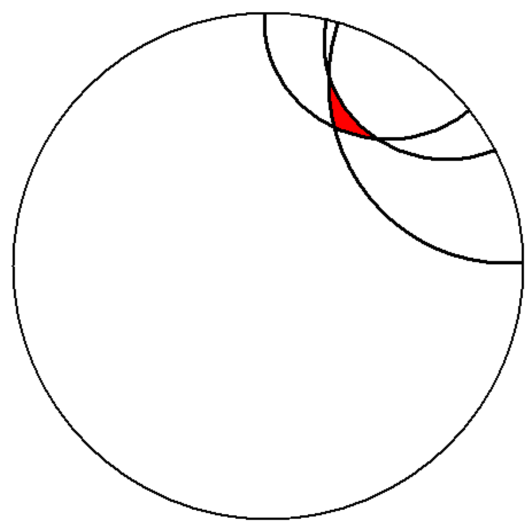
∎

Example (Exercise 1.9.7). Let M be the subset of $\mathbb{R}^2$: $M = \{(u,v) \mid u^2+v^2 < 4k^2\}$ (where $k > 0$). Introduce the metric

$$ds^2 = \frac{1}{\gamma^2}(du^2 + dv^2)$$

where $\gamma(u,v) = 1 - \dfrac{u^2+v^2}{4k^2}$. This is called the *Poincare Disk*. Then $K = -1/k^2$.





Some lines and a triangle in the Poincare disk.

Proof. From Exercise 1.9.4,

$$K = \gamma(\gamma_{uu} + \gamma_{vv}) - (\gamma_u^2 + \gamma_v^2).$$

Well,

$$\gamma_u = \frac{-u}{2k^2}, \gamma_v = \frac{-v}{2k^2}, \gamma_{uu} = \frac{-1}{2k^2}, \gamma_{vv} = \frac{-1}{2k^2}.$$

Therefore,

$$K = \left(1 - \frac{u^2+v^2}{4k^2}\right)\left(\frac{-1}{2k^2} + \frac{-1}{2k^2}\right) - \left(\left(\frac{-u}{2k^2}\right)^2 + \left(\frac{-v}{2k^2}\right)^2\right)$$

$$
\begin{aligned}
&= \left(1 - \frac{u^2 + v^2}{4k^2}\right)\left(\frac{-1}{k^2}\right) - \left(\left(\frac{u^2}{4k^4}\right)^2 + \left(\frac{v^2}{4k^4}\right)^2\right) \\
&= \frac{-(4k^2 - u^2 - v^2)}{4k^4} - \frac{u^2}{4k^4} - \frac{v^2}{4k^4} = \frac{-1}{k^2}.
\end{aligned}
$$

∎

Revised: 6/9/2019