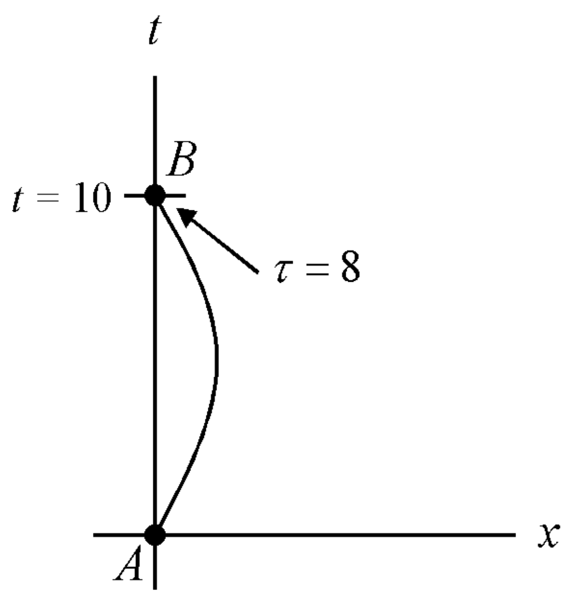
# 2.10 The Twin Paradox

**Note.** Suppose A and B are two events in spacetime separated by a timelike interval (whose $y$ and $z$ coordinates are the same). Joining these events with a straight line produces the world-line of an inertial observer present at both events. Such an observer could view both events as occuring at the same place (say at $x = 0$) and could put these two events along his $t-$axis.

**Note.** Oddly enough, in a spacetime diagram under Lorentz geometry, a straight line gives the longest (spacetime) distance between two points. The figure below illustrates this fact. Here, we have a timelike path and so if we partition the path into little $\Delta\tau$ slices, we have $\Delta\tau = ((\Delta t)^2 - (\Delta x)^2)^{1/2} < \Delta t$.



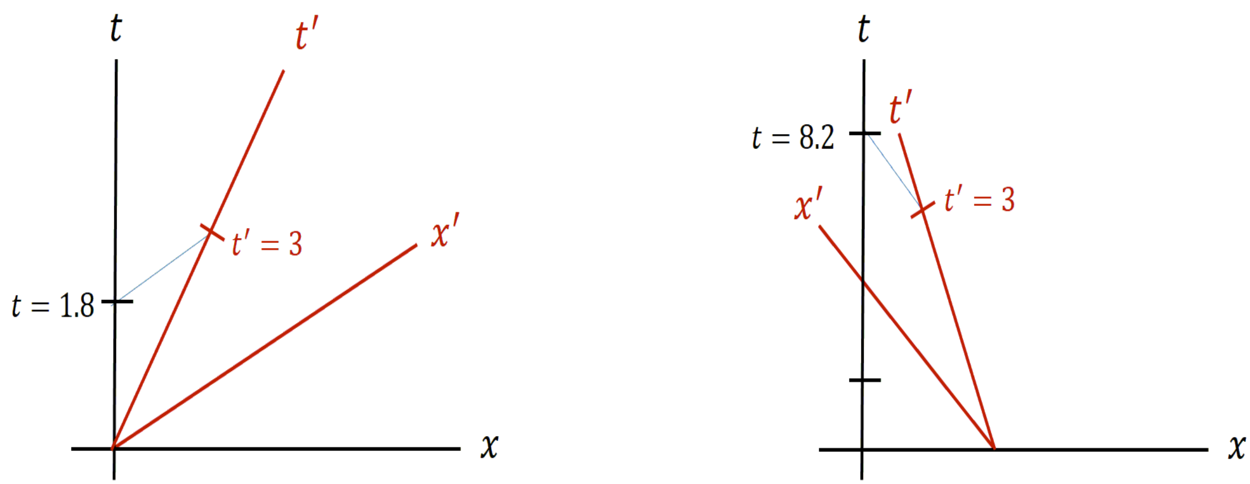

As a consequence, the non-inertial traveler (the one undergoing accelerations and therefore the one not covered by special relativity) from $A$ to $B$ ages less than the inertial traveler between these two events.

**Example (Jack and Jill).** We quote from page 152 of the text: "Let us imagine that Jack is the occupant of a laboratory floating freely in intergalactic space. He can be considered at the origin of an inertial frame of reference. His twin sister, Jill, fires the engines in her rocket, initially alongside Jack's space laboratory. Jill's rocket is accelerated to a speed of 0.8 relative to Jack and then travels at that speed for three years of Jill's time. At the end of that time, Jill fires powerful reversing engines that turn her rocket around and head it back toward Jack's laboratory at the same speed, 0.8. After another three-year period, Jill returns to Jack and slows to a halt beside her brother. Jill is then six years older. We can simplify the analysis by assuming that the three periods of acceleration are so brief as to be negligible. The error introduced is not important, since by making Jill's journey sufficiently long and far, without changing the acceleration intervals, we could make the fraction of time spent in acceleration as small as we wish. Assume Jill travels along Jack's $x-$axis. In Figure II-20 (see below), Jill's world-line is represented on Jack's spacetime diagram. It consists of two straight line segments inclined to the $t-$axis with slopes $+0.8$ and $-0.8$, respectively. For convenience, we are using units of years for time and light-years for distance."

**Note.** Because of the change in direction (necessary to bring Jack and Jill back together), no single inertial frame exists in which Jill is at rest. But her trip can be described in two different inertial frames. Take the first to have $t'$ axis $x = \beta t = 0.8t$ (in Jack's frame). Then at $t = 5$ and $t' = 3$, Jill turns and travels along a new $t'$ axis of $x = -0.8t + 10$ (in Jack's frame). We see that upon the return, Jack has aged 10 years, but Jill has only aged 6 years. This is an example of the *twin paradox*.

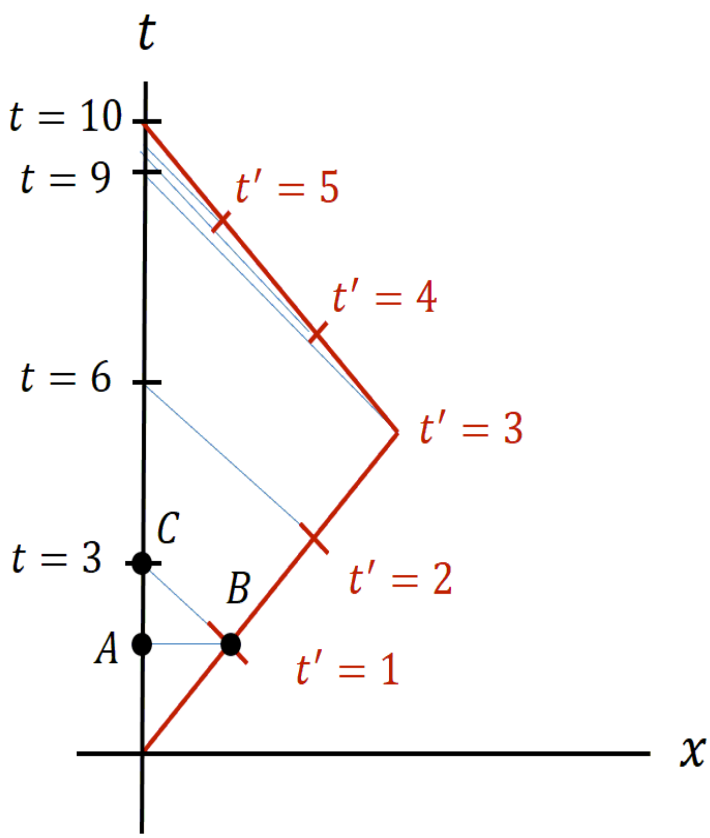
**Note.** One might expect that the Principle of Relativity would imply that Jack should also have aged less than Jill (an obvious contradiction). However, due to the asymmetry of the situation (the fact that Jack is inertial and Jill is not) the Principle of Relativity does not apply.

**Note.** Consider the lines of simultaneity for Jill at the "turning point":




So our assumption that the *effect of Jill's acceleration* is inconsequential is suspect! Jill's "turning" masks a long period of time in Jacks's frame ($t = 1.8$ to $t = 8.2$).

**Note.** Now suppose that Jill emits a flash of light at the end of each (in her frame) year. Consider the spacetime diagram:




The flash of light emitted at event $B$ travels along a 45° line (recall the units) until it intersects the $t-$axis at point $C$ and at time $t_C$. Now distance $AC$ equals distance $AB$ (since $\triangle CBA$ is a 45-45-90). So

$$t_C = t_A + AC = t_A + AB = t_B + x_B.$$

With $t'_B = 1$ and $x'_B = 0$, equation (89) of the Lorentz transformation gives

$$t_B = \frac{\beta x' + t'}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

and

$$x_B = \frac{x' + \beta t'}{\sqrt{1-\beta^2}} = \frac{\beta}{\sqrt{1-\beta^2}}.$$

We therefore have

$$t_C = t_B + x_B = \frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\frac{1+\beta}{\sqrt{1-\beta^2}} = \frac{\sqrt{1+\beta}\sqrt{1+\beta}}{\sqrt{1-\beta}\sqrt{1+\beta}} = \sqrt{\frac{1+\beta}{1-\beta}}.$$

With $\beta = 0.8$ we have $t_C = 3$. Similarly, the second flash is observed by Jack at $t = 6$, and the third flash is observed at $t = 9$. Now the above argument is general and we can show that if a light signal is emitted by Jill every $T$ units of time (in her frame), then Jack receives the signals every $\sqrt{\frac{1+\beta}{1-\beta}}T$ units of time (in her frame). This change in frequency is called the *Doppler effect* and results in a *redshift* (that is, lengthening of wavelength) for $\beta > 0$ and a *blueshift* (that is, a shortening of wavelength) for $\beta < 0$. Notice that the flashes Jill emits at $t' = 3$, $t' = 4$ and $t' = 5$ are observed by Jack at $t = 9$, $t = 9\frac{1}{3}$, and $t = 9\frac{2}{3}$, respectively.

*Revised: 6/30/2019*