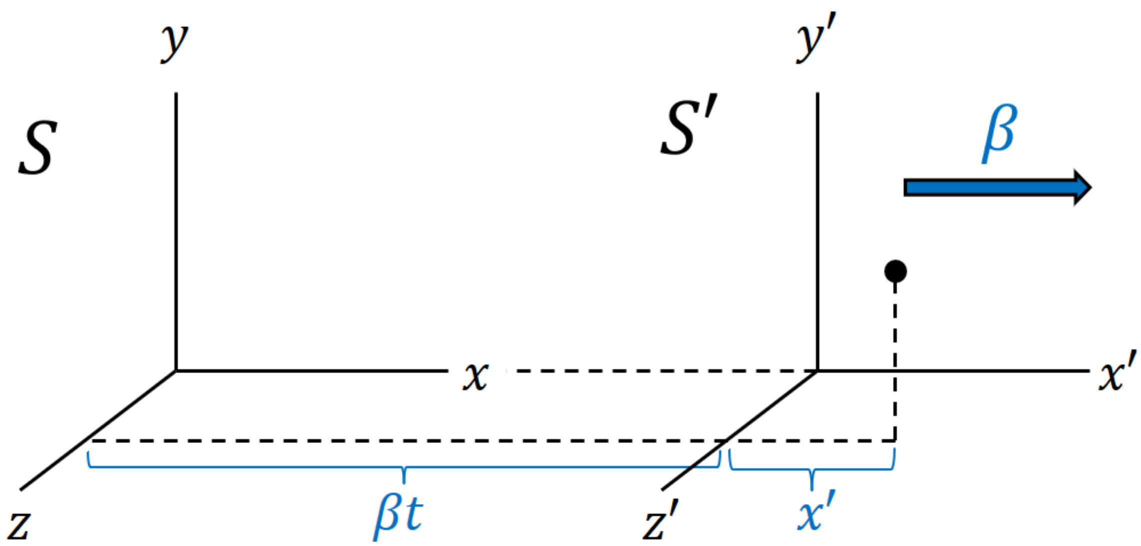
2.7 The Lorentz Transformation

Note. We seek to find the transformation of the coordinates (x, y, z, t) in an inertial frame S to the coordinates (x', y', z', t') in inertial frame S'. Throughout this section, we assume the x and x' axes coincide, S' moves with velocity β in the direction of the positive x axis, and the origins of the systems coincide at $t = t' = 0$.



Note. Classically, we have the relations

$$\begin{aligned} x &= x' + \beta t \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}$$

Definition. The assumption of *homogeneity* says that there is no preferred location in space (that is, space looks the same at all points [on a sufficiently large scale]). The assumption of *isotropy* says that there is no preferred direction in space (that is, space looks the same in every direction).

Note. Under the assumptions of homogeneity and isotropy, the relations between (x, y, z, t) and (x', y', z', t') must be linear (throughout, everything is done in geometric units!):

$$\begin{aligned} x &= a_{11}x' + a_{12}y' + a_{13}z' + a_{14}t' \\ y &= a_{21}x' + a_{22}y' + a_{23}z' + a_{24}t' \\ z &= a_{31}x' + a_{32}y' + a_{33}z' + a_{34}t' \\ t &= a_{41}x' + a_{42}y' + a_{43}z' + a_{44}t'. \end{aligned}$$

If not, say $y = ax'^2$, then a rod lying along the $x-$axis of length $x_b - x_a$ would get longer as we moved it out the $x-$axis, contradicting homogeneity. Similarly, relationships involving time must be linear (since the length of a time interval should not depend on time itself, nor should the length of a spatial interval).

Note. We saw in Section 2.5 that lengths perpendicular to the direction of motion are invariant. Therefore

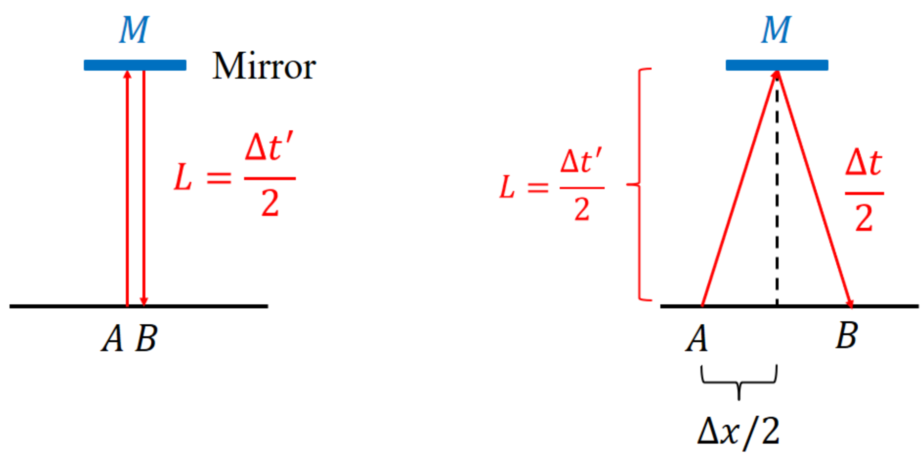
$$\begin{aligned} y &= y' \\ z &= z' \end{aligned}$$

Note. x does not depend on y' and z'. Suppose not. Suppose there is a flat plate at rest in S' and perpendicular to the x'−axis. Since the above equations are linear, an observer in S would see the plate tilted (but still flat) if there is a dependence on y' or z'. However, this implies a "special direction" in space violating the assumption of isotropy. Therefore, the coefficients a_{12} and a_{13} are 0. Similarly, isotropy implies $a_{42} = a_{43} = 0$. We have reduced the system of equations to

$$x = a_{11}x' + a_{14}t' \qquad (85)$$

$$t = a_{41}x' + a_{44}t' \qquad (86)$$

Note. A beam of light is emitted from the origin of S' at time $t = t' = 0$ (when $x = x' = 0$), bounced off a mirror and reflected back to the S' origin ($x' = 0$) at time $t' = \Delta t'$.



In S, the light returns to the origin ($x' = 0$) at time $t = \Delta t = t'/\sqrt{1-\beta^2}$ (equation (78), page 123). Also, in S with $x' = 0$ we have (from equation (86)) that $t = a_{44}t'$. Therefore $a_{44} = 1/\sqrt{1-\beta^2}$. Next, with $x' = 0$ and $t = t'/\sqrt{1-\beta^2}$, since the point $x' = 0$ occurs in the S frame at $x = \beta t$ (due to the relative motion), we have from equation (85):

$$x = \beta t = \frac{\beta}{\sqrt{1-\beta^2}} t' = a_{14} t'$$

and so $a_{14} = \dfrac{\beta}{\sqrt{1-\beta^2}}$. So equations (85) and (86) give

$$x = a_{11} x' + \frac{\beta}{\sqrt{1-\beta^2}} t' \qquad (87)$$

$$t = a_{41} x' + \frac{1}{\sqrt{1-\beta^2}} t' \qquad (88)$$

Note. Now consider a flash of light emitted at the origins of S and S' at $t = t' = 0$. This produces a sphere of light in each frame (according to the constancy of the speed of light). And so

$$t^2 = x^2 + y^2 + z^2$$

$$t'^2 = x'^2 + y'^2 + z'^2.$$

Since $y = y'$ and $z = z'$, we have $t^2 - x^2 = t'^2 - x'^2$. From equations (87) and (88) we get

$$t^2 - x^2 = \left(a_{41} x' + \frac{1}{\sqrt{1-\beta^2}} t' \right)^2 - \left(a_{11} x' + \frac{\beta}{\sqrt{1-\beta^2}} t' \right)^2 = t'^2 - x'^2.$$

Expanding

$$(a_{41})^2 x'^2 + \left(\frac{2a_{41}}{\sqrt{1-\beta^2}} \right) x't' + \frac{1}{1-\beta^2} t'^2 - (a_{11})^2 x'^2$$

$$-2\frac{a_{11}\beta}{\sqrt{1-\beta^2}} x't' - \frac{\beta^2}{1-\beta^2} t'^2 = t'^2 - x'^2$$

or

$$x'^2 (a_{41}^2 - a_{11}^2) + x't' \left(\frac{2a_{41}}{\sqrt{1-\beta^2}} - 2\frac{a_{11}\beta}{\sqrt{1-\beta^2}} \right)$$

$$+t'^2\left(\frac{1}{1-\beta^2}-\frac{\beta}{1-\beta^2}\right)=t'^2-x'^2.$$

Comparing coefficients, we need

$$a_{41}^2-a_{11}^2=-1$$

$$2\frac{a_{41}}{\sqrt{1-\beta^2}}-2\frac{a_{11}\beta}{\sqrt{1-\beta^2}}=0$$

or

$$a_{41}^2-a_{11}^2=-1 \quad \text{and} \quad a_{41}-\beta a_{11}=0.$$

Solving this system: $a_{41}=\beta a_{11}$ and so

$$a_{41}^2-a_{11}^2=(\beta a_{11})^2-a_{11}^2=-1$$

or

$$a_{11}^2=\frac{1}{1-\beta^2} \quad \text{and} \quad a_{11}=\pm\frac{1}{\sqrt{1-\beta^2}}.$$

From equation (87) with $\beta=0$ we see that $x=\left(a_{11}|_{\beta=0}\right)x'$ and we want $x=x'$ in the event that $\beta=0$. Therefore, we have

$$a_{11}=\frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad a_{41}=\frac{\beta}{\sqrt{1-\beta^2}}.$$

We now have the desired relations between (x,y,z,t) and (x',y',z',t').

Definition. The transformation relating coordinates (x, y, z, t) in S to coordinates (x', y', z', t') in S' given by

$$\begin{aligned} x &= \frac{x' + \beta t'}{\sqrt{1-\beta^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{\beta x' + t'}{\sqrt{1-\beta^2}} \end{aligned}$$

is called the *Lorentz Transformation.*

Note. With $\beta \ll 1$ and $\beta^2 \approx 0$ we have

$$\begin{aligned} x &= x' + \beta t' \\ t &= t' \end{aligned}$$

(in geometric units, x and x' are small compared to t and t' [see page 117; remember time gets multiplied by c to express it in units of length] and $\beta x'$ is negligible compared to t', but $\beta t'$ is NOT negligible compared to x').

Note. By the Principle of Relativity, we can invert the Lorentz Transformation simply by interchanging x and t with x' and t', respectively, and replacing β with $-\beta$!

Note. If we deal with pairs of events separated in space and time, we denote the differences in coordinates with Δ's to get

$$\Delta x = \frac{\Delta x' + \beta \Delta t'}{\sqrt{1-\beta^2}} \qquad (91a)$$

$$\Delta t = \frac{\beta \Delta x' + \Delta t'}{\sqrt{1-\beta^2}} \qquad (91b)$$

With $\Delta x' = 0$ in (91b) we get the equation for time dilation. With a rod of length $L = \Delta x$ in frame S, the length measured in S' requires a simultaneous measurement of the endpoints ($\Delta t' = 0$) and so from (91a) $L = L'/\sqrt{1-\beta^2}$ or $L' = L\sqrt{1-\beta^2}$, the equation for length contraction.

Example (Exercise 2.7.2). Observer S' seated at the center of a railroad car observes two men, seated at opposite ends of the car, light cigarettes simultaneously ($\Delta t' = 0$). However for S, an observer on the station platform, these events are not simultaneous ($\Delta t \neq 0$). If the length of the railroad car is $\Delta x' = 25$m and the speed of the car relative to the platform is 20m/sec ($\beta = 20/3 \times 10^8$), find Δt and convert your answer to seconds.

Solution. We have $\Delta t' = 0$, $\Delta x' = 25$m, and $\beta = 20/3 \times 10^8 \approx 6.67 \times 10^{-8}$. So by equation (91b)

$$\Delta t = \frac{\beta \Delta x' + \Delta t'}{\sqrt{1-\beta^2}} = \frac{(6.67 \times 10^{-8})(25\text{m})}{\sqrt{1-(6.67 \times 10^{-8})^2}} \approx 1.67 \times 10^{-6}\text{m}$$

or in seconds

$$\Delta t = \frac{1.67 \times 10^{-6}\text{m}}{3 \times 10^8\text{m/sec}} = 5.56 \times 10^{-15}\text{sec}.$$

Example (Exercise 2.7.14). Substitute the transformation Equation (91) into the formula for the interval and verify that

$$(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2.$$

Solution. With $\Delta y = \Delta z = 0$ we have

$$\begin{aligned}
(\Delta t)^2 & -(\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\
= & \quad (\Delta t)^2 - (\Delta x)^2 = \left(\frac{\beta \Delta x' + \Delta t'}{\sqrt{1-\beta^2}}\right)^2 - \left(\frac{\Delta x' + \beta \Delta t'}{\sqrt{1-\beta^2}}\right)^2 \\
= & \quad \frac{\beta^2(\Delta x')^2 + 2\beta \Delta x' \Delta t' + (\Delta t')^2 - (\Delta x')^2 - 2\beta \Delta x' \Delta t' - \beta^2 (\Delta t')^2}{1-\beta^2} \\
= & \quad \frac{(\Delta x')^2(\beta^2 - 1) + (\Delta t')^2(1-\beta^2)}{1-\beta^2} \\
= & \quad (\Delta t')^2 - (\Delta x')^2 = (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2
\end{aligned}$$

since $\Delta y' = \Delta z' = 0$.

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