2.9 Lorentz Geometry

Note. We wish to extend the idea of arclength to 4-dimensional spacetime. We do so by replacing the idea of "distance" ($\sqrt{\sum(\Delta x^i)^2}$) by the interval. The resulting geometry is called *Lorentz geometry*.

Definition. Let $\vec{\alpha}$ be a curve in spacetime. The *spacetime length* (or *proper time*) of $\vec{\alpha}$ is

$$L(\alpha) = \int_\alpha d\tau = \int_\alpha \sqrt{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}.$$

Note. Since $\Delta\tau$ (and so $d\tau$) is an invariant from one inertial frame to another, then so is $L(\alpha)$. $L(\alpha)$ may be viewed as the actual passage of time that would be recorded for a clock with world-line $\vec{\alpha}$ (this is certainly clear when $dx = dy = dz = 0$).

Definition. $\mathbb{R}^4$ with the semi-Riemannian metric

$$d\tau^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \qquad (93)$$

is called *Minkowski space*.

Note. It may seem a bit unusual to see $(d\tau)^2$ referred to as the "metric," but of course it does determine a way to measure the distance between points—although the "distance" may be negative. In a tensor analysis setting, a *metric tensor* is a bilinear form which is non-degenerate (and may sometimes be "positive definite" and sometimes "negative definite"; that is, we may sometimes get negative distances from a metric tensor). For more details on this behavior, see my online notes based on Dodson and Poston's *Tensor Geometry* (Springer-Verlag, 1991)on Section IV.1. Metrics.

Note. We parameterized curves with respect to arclength in Chapter 1. It is convenient to parameterize timelike curves (those curves for which $(d\tau/dt)^2 > 0$) in terms of proper time.

Example. Suppose a free particle travels with constant speed and direction, so that

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c$$

for constants a, b, c. Define $\beta = \sqrt{a^2 + b^2 + c^2}$ (the particle's speed). From equation (93),

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2 = 1 - \beta^2.$$

Since $d\tau/dt$ is constant, τ is a monotone function of t and so $dt/d\tau = 1/\sqrt{1-\beta^2}$ (well $\pm$)

$$\begin{aligned}\frac{dx}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} = \frac{a}{\sqrt{1-\beta^2}}\\ \frac{dy}{d\tau} &= \frac{dy}{dt}\frac{dt}{d\tau} = \frac{b}{\sqrt{1-\beta^2}}\\ \frac{dz}{d\tau} &= \frac{dz}{dt}\frac{dt}{d\tau} = \frac{c}{\sqrt{1-\beta^2}}.\end{aligned}$$

Notice that each of these derivatives is constant and so the particle follows a straight line in spacetime. If we calculate second derivatives, we see that a free particle satisfies:

$$\frac{d^2t}{d\tau^2} = \frac{d^2x}{d\tau^2} = \frac{d^2y}{d\tau^2} = \frac{d^2z}{d\tau^2} = 0.$$

In fact, free particles follow geodesics in the spacetime of special relativity (in which geodesics are straight lines).

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