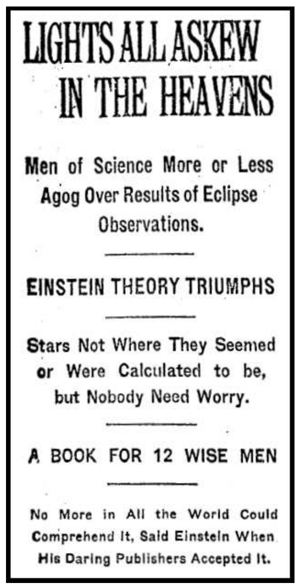
3.10 The Bending of Light

Note. In this section, we address the bending of light mentioned in Section 3.3 (which leads to the Einstein ring photos). It was the confirmation of Einstein's prediction (made in his 1916 paper) by Arthur Eddington's eclipse expedition to the island of Príncipe off the coast of West Africa on May 29, 1919 which lead to much of the media attention which Einstein was to get in the 1920s and later. In fact, an announcement of the results of Eddington's expedition appeared in the November 10, 1919 *New York Times*:


LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Note. A photon of light follows a lightlike geodesic for which $d\tau = 0$. Such a geodesic, therefore, cannot be parameterized in terms of τ. So we parameterize in terms of some ρ where $d\tau/d\rho = 0$. As in the previous section, let the geodesic be $(x^0(\rho), x^1(\rho), x^2(\rho), x(^3(\rho))$ and we have:

$$\frac{d^2x^\lambda}{d\rho^2} + \Gamma^\lambda_{\mu\nu}\frac{d^\mu}{d\rho}\frac{dx^\nu}{d\rho} = 0$$

for $\lambda = 0, 1, 2, 3$.

Note. As in the previous section, we desire equations (159a–c) with τ replaced with ρ (again, we assume the photon is restricted to the plane $\varphi = \pi/2$).

Note. As in the previous section

$$d\tau^2 \;=\; \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\varphi^2 - r^2 \sin^2\varphi\, d\theta^2$$

(this is equation (157)) and so

$$d\tau^2 = \gamma dt^2 - \gamma^{-1} dr^2 - r^2 d\theta^2$$

(since $\varphi = \pi/2$ and $d\varphi = 0$) and

$$\left(\frac{d\tau^2}{d\rho^2}\right) = \gamma \left(\frac{dt}{d\rho}\right)^2 - \gamma^{-1} \left(\frac{dr}{d\rho}\right)^{-1} - r^2 \left(\frac{d\theta}{d\rho}\right)^2$$

where $\gamma = 1 - 2M/r$. Now $d\tau/d\rho = 0$ so with the notation of the previous section (see equation (162)) we have

$$0 = \gamma \left(\frac{b}{\gamma}\right)^2 - \gamma^{-1} \left(\frac{dr}{d\theta}\frac{h}{r^2}\right)^2 - r^2 \left(\frac{h}{r^2}\right)^2$$

or

$$0 = b^2 - \left(\frac{dr}{d\theta}\frac{h}{r^2}\right)^2 - \gamma r^2 \left(\frac{h}{r^2}\right)^2$$

or

$$\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + \gamma \frac{h^2}{r^2} = b^2.$$

Now (as on page 226) with $u = 1/r$ and $\dfrac{du}{d\theta} = \dfrac{-1}{r^2}\dfrac{dr}{d\theta}$ we get

$$h^2 \left(\frac{du}{d\theta}\right)^2 + \left(1 - \frac{2M}{r}\right) h^2 u^2 = b^2$$

or

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{b^2}{h^2} + \frac{2Mu^2}{r} = \frac{b^2}{h^2} + 2Mu^3.$$

Differentiating with respect to θ implies

$$2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = 0 + 6Mu^2\frac{du}{d\theta}$$

or (dividing by $2du/d\theta$):

$$\frac{d^2u}{d\theta^2} + u = 3Mu^2 \qquad (168)$$

(compare to (163)).

Note. We orient our coordinate system such that the closest approach of the geodesic occurs at $\theta = 0$. If $M = 0$, the general solution to (168) (a homogeneous equation under this condition) is $u = \alpha\cos\theta + \beta\sin\theta$. If we let R be the minimum distance of the geodesic from M (assumed to occur at $\theta = 0$) we have $\alpha = 1/R$. With $M = 0$, geodesics are "straight lines" and so $\beta = 0$ and the resulting homogeneous solution is $u_h = (1/R)\cos\theta$.

Note. We modify (168) by substituting $u_h = (1/R)\cos\theta$ in the right hand side to produce (as in the previous section)

$$\begin{aligned}\frac{d^2u}{d\theta^2} + u \approx 3M\left(\frac{1}{R}\cos\theta\right)^2 &= \frac{3M}{R^2}\cos^2\theta \\ &= \frac{3M}{2R^2}\left(1+\cos(2\theta)\right).\end{aligned}$$

By Lemma III-5, a particular solution is

$$u_p \approx \frac{3M}{2R^2}\left(1 - \frac{1}{3}\cos(2\theta)\right)$$

or (since $\cos(2\theta) = 2\cos^2\theta - 1$)):

$$u_p \approx \frac{M}{R^2}(2 - \cos^2\theta).$$

Now adding the homogeneous solution $u = (1/R)\cos\theta$ to this particular solution we get

$$u = \frac{1}{R} \approx u_p + u_h = \frac{M}{R^2}(2 - \cos^2\theta) + \frac{1}{R}\cos\theta. \quad (170)$$

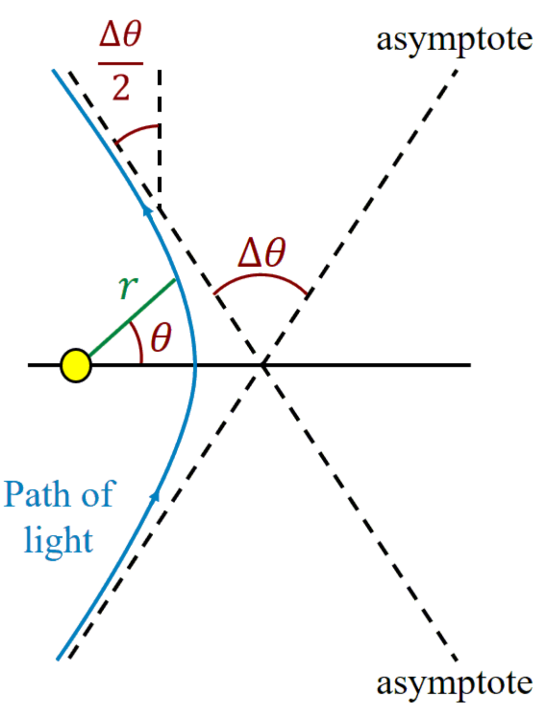
Note. From the figure below, we have for $r \to \infty$, θ approaches $\pm(\pi/2 + \Delta\theta/2)$. Since $\Delta\theta \approx 0$, $\cos^2\theta \approx 0$ and so for $r \to \infty$ we have from (170) that

$$\begin{aligned} u &= \frac{1}{r} \approx 0 \approx \frac{1}{R}\cos\theta + \frac{M}{R^2}(2 - \cos^2\theta) \\ &\to \frac{1}{R}\cos\left(\frac{\pi}{2} + \frac{\Delta\theta}{2}\right) + \frac{2M}{R^2} - 0. \end{aligned}$$

Or

$$\frac{2M}{R^2} \approx \frac{-1}{R}\cos\left(\frac{\pi}{2} + \frac{\Delta\theta}{2}\right) = \frac{1}{R}\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2R}.$$

Therefore $\dfrac{\Delta\theta}{2} \approx \dfrac{2M}{R}$ and $\Delta\theta \approx \dfrac{4M}{R}$.



Note. If we assume a photon undergoes a Newtonian acceleration, we find that Newtonian mechanics implies that a photon follows a hyperbolic trajectory with $\Delta\theta = \dfrac{2M_\odot}{R_\odot}$ when grazing the Sun. This is half the displacement predicted by general relativity.

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