3.6 Geodesics

Note. We now view spacetime as a semi-Riemannian 4-manifold such that for each coordinate system (x^0, x^1, x^2, x^3)

$$(d\tau)^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where the g_{ij} are functions of the coordinates.

Definition. A vector $\vec{v} = v^\mu \frac{\partial}{\partial x^\mu}$ is *timelike*, *lightlike*, or *spacelike* if $\langle \vec{v}, \vec{v} \rangle = g_{\mu\nu} v^\mu v^\nu$ is positive, zero, or negative, respectively.

Definition. A spacetime curve $\vec{\alpha}$ is a *geodesic* if it has a parameterization $x^\lambda(\rho)$ satisfying

$$\frac{d^2 x^\lambda}{d\rho^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\rho} \frac{dx^\nu}{d\rho} = 0 \qquad (120)$$

for $\lambda = 0, 1, 2, 3$.

Note. "It can be shown" that the definition of geodesic is independent of a choice of coordinate system (says the text, page 198).

Note. If $\vec{\alpha}$ is a geodesic, then

$$\langle \vec{\alpha}', \vec{\alpha}' \rangle = \left(\frac{d\tau}{d\rho} \right)^2 = g_{\mu\nu} \frac{dx^\mu}{d\rho} \frac{dx^\nu}{d\rho}$$

is constant (with the same proof as given at the bottom of page 68).

Definition. A geodesic $\vec{\alpha}$ is *timelike*, *lightlike*, or *spacelike* according to whether $\langle \vec{\alpha}', \vec{\alpha}' \rangle$ is positive, zero, or negative.

Note. If a geodesic $\vec{\alpha}$ is timelike, then $d\tau/d\rho =$ constant, and we have $\rho = a\tau + b$ for some a and b. We then see that equation (120) still holds when we replace ρ with τ.

Note. If a geodesic $\vec{\alpha}$ is lightlike then

$$\langle \vec{\alpha}', \vec{\alpha}' \rangle = \left(\frac{d\tau}{d\rho} \right)^2 = 0$$

and τ is constant along $\vec{\alpha}$. Therefore proper time τ cannot be used to reparameterize $\vec{\alpha}$.

Note. If a geodesic $\vec{\alpha}$ is spacelike, then $d\tau/d\rho$ is imaginary. The proper distance

$$d\sigma = \sqrt{(dx)^2 + (dy)^2 + (dz)^2 - (dt)^2} = i d\tau$$

can be used to parameterize $\vec{\alpha}$. We have $d\sigma/d\rho$ a real constant and so $\rho = a\sigma + b$ for some a and b. We see that equation (120) still holds when we replace ρ with σ.

Definition. A curve $\vec{\alpha}$ is *timelike* if $\langle \vec{\alpha}', \vec{\alpha}' \rangle > 0$ at each of its points.

Theorem III-2. Let $\vec{\alpha}$ be a timelike curve which extremizes spacetime distance (i.e. the quantity $\Delta\tau$) between its two end points. Then $\vec{\alpha}$ is a geodesic.

Idea of the proof. The curve can be parameterized in terms of τ (as remarked above). The proof then follows as did the proof of Theorem I-9.

Theorem III-3. Given an event $\vec{P}$ and a nonzero vector $\vec{v}$ at $\vec{P}$, then there exists a unique geodesic $\vec{\alpha}$ such that $\vec{\alpha}(0) = \vec{P}$ and $\vec{\alpha}'(0) = \vec{v}$.

Note. Theorem III-3 implies that all particles in a gravitational field will fall with the same acceleration dependent only on initial position and velocity. That is, we don't see heavier objects fall faster!

Note. In the absence of gravity, a particle follows a path $\frac{d^2x^\lambda}{d\rho^2} = 0$ (that is, the particle follows a straight line!). We can therefore interpret the Christoffel symbols as the components of the gravitational field.

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